An introduction to Topological Quantum Computing

Shawn Xingshan Cui
Dept. of Mathematics
Dept. of Physics and Astronomy
Purdue University

Mini course for
Algebraic structures in topology 2026

University of Puerto Rico, Río Piedras, July 10-17, 2026

1

(physics)
topological phase of matter

topological
quantum computing

2

(algebra)
higher category theory

3

(topology)
topological quantum field theory

Part I: Outline

- Basics of quantum computing/mechanics
- Motivations to topological quantum computing (TQC) and what it is
- Anyons and topological phases of matter from lattice models

Axiomatic Formulation of Quantum Mechanics

★ Quantum states

- A quantum system is associated with a state/Hilbert space $(\mathscr{H}, \langle \cdot | \cdot \rangle)$
- At each time t, the system is described by a state $|\psi(t)\rangle \in \mathscr{H}$, which is a unit vector, defined up to scalar multiplication

Axiomatic Formulation of Quantum Mechanics

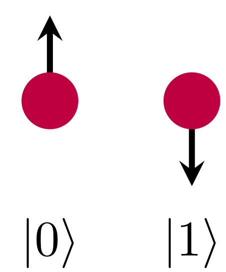
★ Quantum states

- A quantum system is associated with a state/Hilbert space ($\mathscr{H}$, $\langle \cdot | \cdot \rangle$)
- At each time t, the system is described by a state $|\psi(t)\rangle \in \mathscr{H}$, which is a unit vector, defined up to scalar multiplication

Example: an electron (only consider its spin degrees)

State space: $\mathbb{C}^2$

'spin up' state: $|0\rangle$



'spin down' state: $|1\rangle$

A state at superposition: $(|0\rangle - |1\rangle)/\sqrt{2}$

A general state: $\alpha|0\rangle + \beta|1\rangle$

★ Time evolution

- The state obeys the **Schrodinger Equation**:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$\hat{H}$: Hamiltonian; Hermitian operator acting on $\mathscr{H}$

★ Time evolution

- The state obeys the **Schrodinger Equation**:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$\hat{H}$: Hamiltonian; Hermitian operator acting on $\mathscr{H}$

Exercise: if $\hat{H}$ is constant, then $|\psi(t)\rangle = \exp(-it\hat{H})|\psi(0)\rangle$

unitary operator

★ Time evolution

- The state obeys the **Schrodinger Equation**:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$\hat{H}$: Hamiltonian; Hermitian operator acting on $\mathscr{H}$

Exercise: if $\hat{H}$ is constant, then $|\psi(t)\rangle = \exp(-it\hat{H})|\psi(0)\rangle$

unitary operator

- Equivalently, there exists unitary operators $U(t)$, such that

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

★ Measurement

- A measurable quantity (e.g. spin, momentum, position) corresponds to a Hermitian operator $\hat{Q}$ acting on $\mathscr{H}$

λ : eigenvalue of $\hat{Q}$

P_λ : orthogonal projection onto the λ-eigenspace

$$\hat{Q} = \sum_\lambda \lambda \, P_\lambda$$

★ Measurement

- A measurable quantity (e.g. spin, momentum, position) corresponds to a Hermitian operator $\hat{Q}$ acting on $\mathscr{H}$

  λ : eigenvalue of $\hat{Q}$

  P_λ : orthogonal projection onto the λ-eigenspace

$$\hat{Q} = \sum_\lambda \lambda \; P_\lambda$$

- If a system is at state $|\psi\rangle$, then measuring $\hat{Q}$ changes the state to $P_\lambda|\psi\rangle$ with probability $\langle\psi|P_\lambda|\psi\rangle$

★ Measurement

- A measurable quantity (e.g. spin, momentum, position) corresponds to a Hermitian operator $\hat{Q}$ acting on $\mathscr{H}$

  λ : eigenvalue of $\hat{Q}$

  P_λ : orthogonal projection onto the λ-eigenspace

$$\hat{Q} = \sum_\lambda \lambda \, P_\lambda$$

- If a system is at state $|\psi\rangle$, then measuring $\hat{Q}$ changes the state to $P_\lambda|\psi\rangle$ with probability $\langle\psi|P_\lambda|\psi\rangle$

Example: electron spin $\mathbb{C}^2 = \mathrm{span}\{|0\rangle, |1\rangle\}$

$$\hat{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{Q}|0\rangle = |0\rangle$$

$$\hat{Q}|1\rangle = -|1\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |0\rangle, \ \mathrm{Prob} = |\alpha|^2$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |1\rangle, \ \mathrm{Prob} = |\beta|^2$$

★ Composite system

If two systems have state spaces $\mathscr{H}_1$, $\mathscr{H}_2$, then their joint system has state space $\mathscr{H}_1 \otimes \mathscr{H}_2$.

★ Composite system

If two systems have state spaces $\mathscr{H}_1$, $\mathscr{H}_2$, then their joint system has state space $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Example: 2-electron spins $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$|0\rangle \otimes |1\rangle \qquad\qquad \big(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\big)/\sqrt{2} \;\neq\; |\psi_1\rangle \otimes \psi_2\rangle$$

product state

entangled state

★ Composite system

If two systems have state spaces $\mathscr{H}_1$, $\mathscr{H}_2$, then their joint system has state space $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Example: 2-electron spins $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$|0\rangle \otimes |1\rangle \qquad\qquad \big(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\big)/\sqrt{2} \neq |\psi_1\rangle \otimes \psi_2\rangle$$

product state   entangled state

★ Identical particles

For N identical particles with state space $\mathscr{H}_{tot} := \mathscr{H} \otimes \cdots \otimes \mathscr{H}$, where $\mathscr{H}$ is the single-particle space. The permutation group S_N acts on $\mathscr{H}_{tot}$:

$$\tau_{ij}|\psi_1\rangle \otimes \cdots |\psi_i\rangle \cdots |\psi_j\rangle \cdots \otimes |\psi_n\rangle = |\psi_1\rangle \otimes \cdots |\psi_j\rangle \cdots |\psi_i\rangle \cdots \otimes |\psi_n\rangle$$

★ Composite system

If two systems have state spaces $\mathscr{H}_1$, $\mathscr{H}_2$, then their joint system has state space $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Example: 2-electron spins $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$|0\rangle \otimes |1\rangle \qquad \qquad \big(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\big)/\sqrt{2} \;\neq\; |\psi_1\rangle \otimes \psi_2\rangle$$

product state   entangled state

★ Identical particles

For N identical particles with state space $\mathscr{H}_{tot} := \mathscr{H} \otimes \cdots \otimes \mathscr{H}$, where $\mathscr{H}$ is the single-particle space. The permutation group S_N acts on $\mathscr{H}_{tot}$:

$$\tau_{ij}|\psi_1\rangle \otimes \cdots |\psi_i\rangle \cdots |\psi_j\rangle \cdots \otimes |\psi_n\rangle = |\psi_1\rangle \otimes \cdots |\psi_j\rangle \cdots |\psi_i\rangle \cdots \otimes |\psi_n\rangle$$

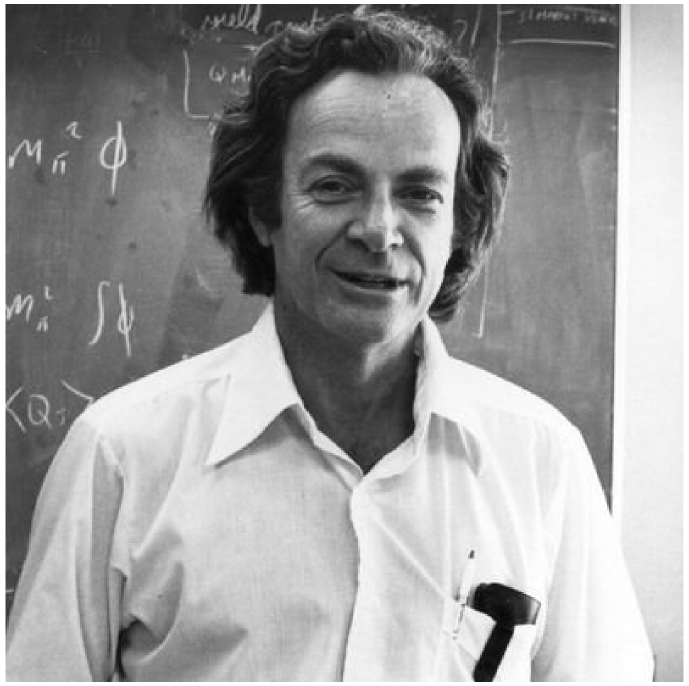
A multi-particle state is

either symmetric, i.e. $\tau_{ij}|\psi\rangle = |\psi\rangle, \forall i, j \quad \Longrightarrow$ bosons

or anti-symmetric, i.e. $\tau_{ij}|\psi\rangle = -|\psi\rangle, \forall i, j \quad \Longrightarrow$ fermions

Origin of Quantum Computing

Early days: ’80 - ’90



Feynman

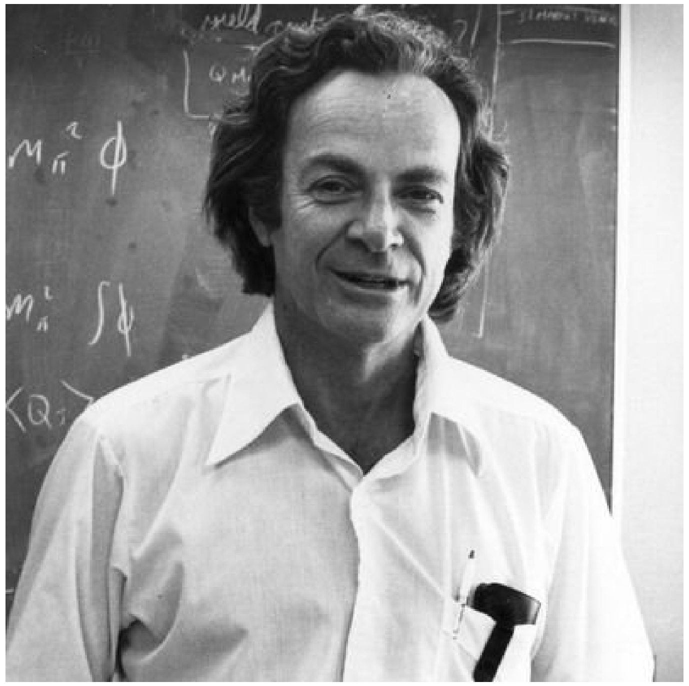
Manin

$$i\hbar\frac{\partial}{\partial t}|\psi(x,t)\rangle = \hat{H}|\psi(x,t)\rangle$$

Quantum Mechanics

Origin of Quantum Computing

Early days: ’80 - ’90



Feynman

Manin

Deutsch

Shor

$$i\hbar\frac{\partial}{\partial t}|\psi(x,t)\rangle = \hat{H}|\psi(x,t)\rangle$$

Quantum Mechanics

RSA-260

2211282552952966643528108525502623092761208950247001
5394413748319128822941402001986512729726569746599085
9003300314000511707422045608592763579537571859542988
3895870922923849100670303412462054578456641366454068
4214361293017694020846391065875914794251435144458199

= (?) x (?)

Classical computation

★ Bit: {0, 1}

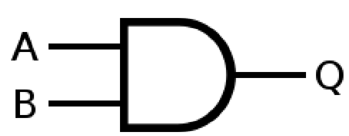
State/bit string: 0011010101111

Classical computation

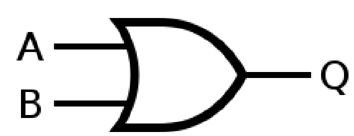
★ Bit: {0, 1}

State/bit string: 0011010101111

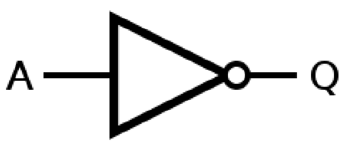
★ Logic gates:



$Q = AB$



$Q = A \oplus B \oplus AB$



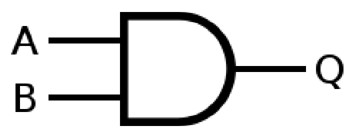
$Q = 1 \oplus A$

Classical computation

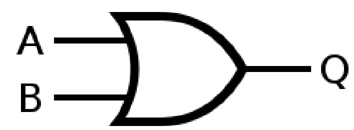
★ Bit: {0, 1}

State/bit string: 0011010101111

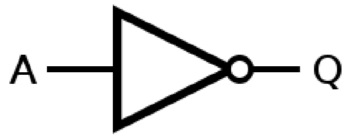
★ Logic gates:



$Q = AB$

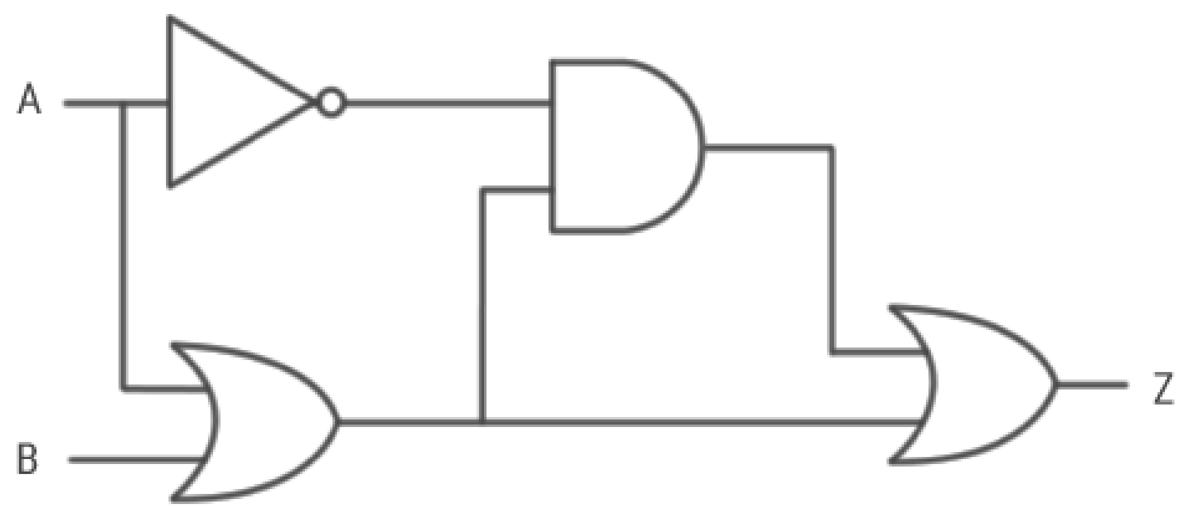


$Q = A \oplus B \oplus AB$



$Q = 1 \oplus A$

Process of computing:



https://www.computerscience.gcse.guru/theory/logic-circuits

Quantum computation

★ Qubit: 2-dim state space $\mathbb{C}^2$ with an orthonormal basis $\{\,|0\rangle, |1\rangle\,\}$

Multiple qubits: $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

Quantum states, e.g.

$$|0\rangle \qquad |1\rangle \qquad |0\rangle \otimes |1\rangle \otimes \cdots \otimes |0\rangle \equiv |01\cdots 0\rangle$$

$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

Quantum computation

★ Qubit: 2-dim state space $\mathbb{C}^2$ with an orthonormal basis $\{\,|\,0\rangle, |\,1\rangle\,\}$

Multiple qubits: $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

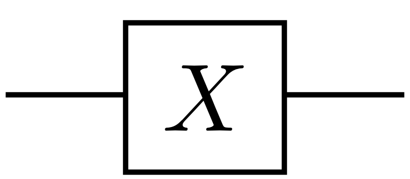
Quantum states, e.g.

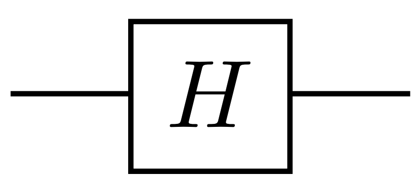
$$|0\rangle \qquad |1\rangle \qquad |0\rangle \otimes |1\rangle \otimes \cdots \otimes |0\rangle \equiv |01\cdots 0\rangle$$

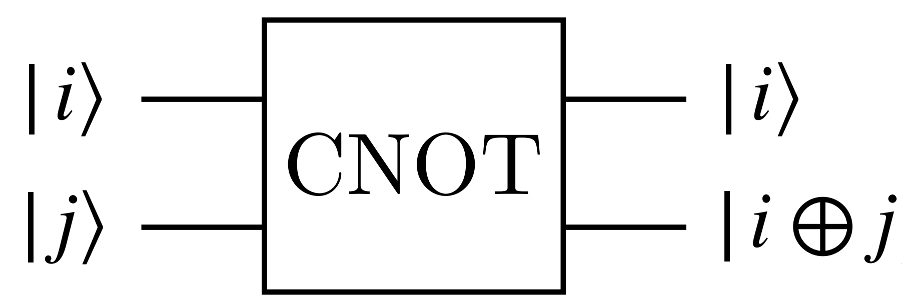
$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

★ Quantum gates: unitary transformations on $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \mathrm{CNOT} \colon \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$$







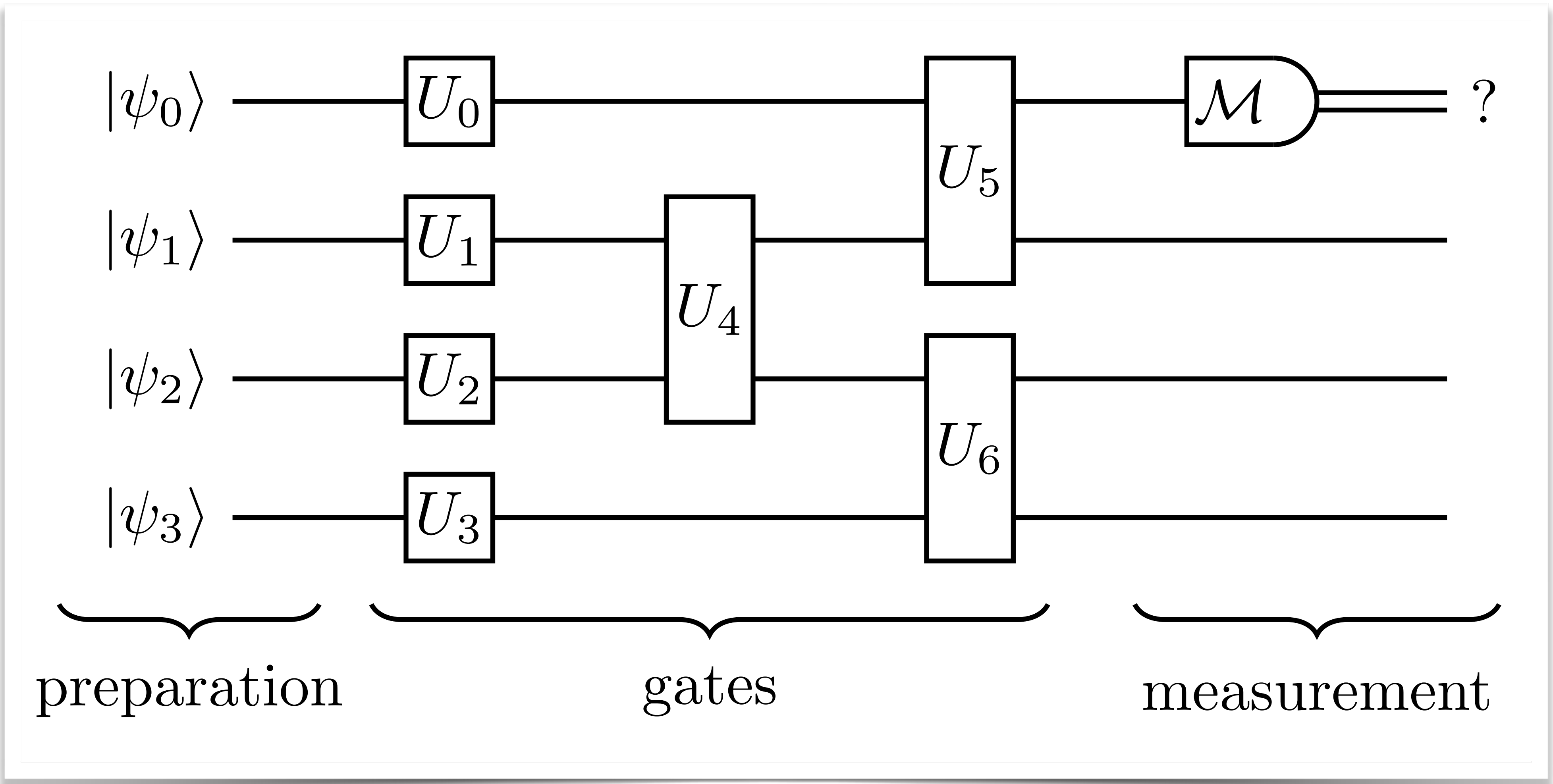
★ Measurement: a way to “see” quantum states.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \begin{array}{l} \nearrow \;\; |0\rangle,\ \text{Prob}= |\alpha|^2 \\ \\ \searrow \;\; |1\rangle,\ \text{Prob}= |\beta|^2 \end{array}$$

★ Measurement: a way to “see” quantum states.

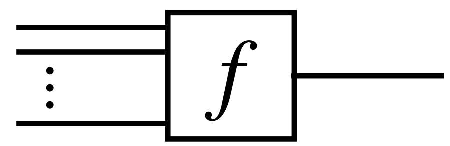
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \begin{cases} |0\rangle, \ \text{Prob}= |\alpha|^2 \\ |1\rangle, \ \text{Prob}= |\beta|^2 \end{cases}$$

Process of quantum computing:



Example: Deutsch–Jozsa algorithm

Given a blackbox implementing a function $f:\{0,1\}^n \to \{0,1\}$,

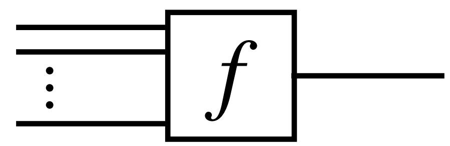


It's assured f is either constant, or balanced, i.e. $|f^{-1}(0)| = |f^{-1}(1)|$.

How many times in the worst case to call the blackbox to determine the type of f?

Example: Deutsch–Jozsa algorithm

Given a blackbox implementing a function $f:\{0,1\}^n \to \{0,1\}$,



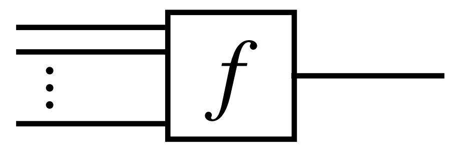
It's assured f is either constant, or balanced, i.e. $|f^{-1}(0)| = |f^{-1}(1)|$.

How many times in the worst case to call the blackbox to determine the type of f?

- Classically: $2^{n-1}+1$

Example: Deutsch–Jozsa algorithm

Given a blackbox implementing a function $f: \{0,1\}^n \to \{0,1\}$,

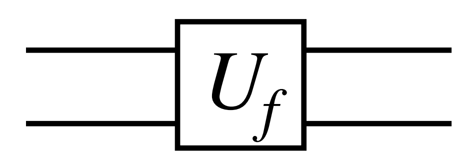


It's assured f is either constant, or balanced, i.e. $|f^{-1}(0)| = |f^{-1}(1)|$.

How many times in the worst case to call the blackbox to determine the type of f ?

- Classically: $2^{n-1} + 1$
- Quantum: ?

  Consider $n = 1$. Let $U_f|x, y\rangle = |x, u \oplus f(x)\rangle$,



$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

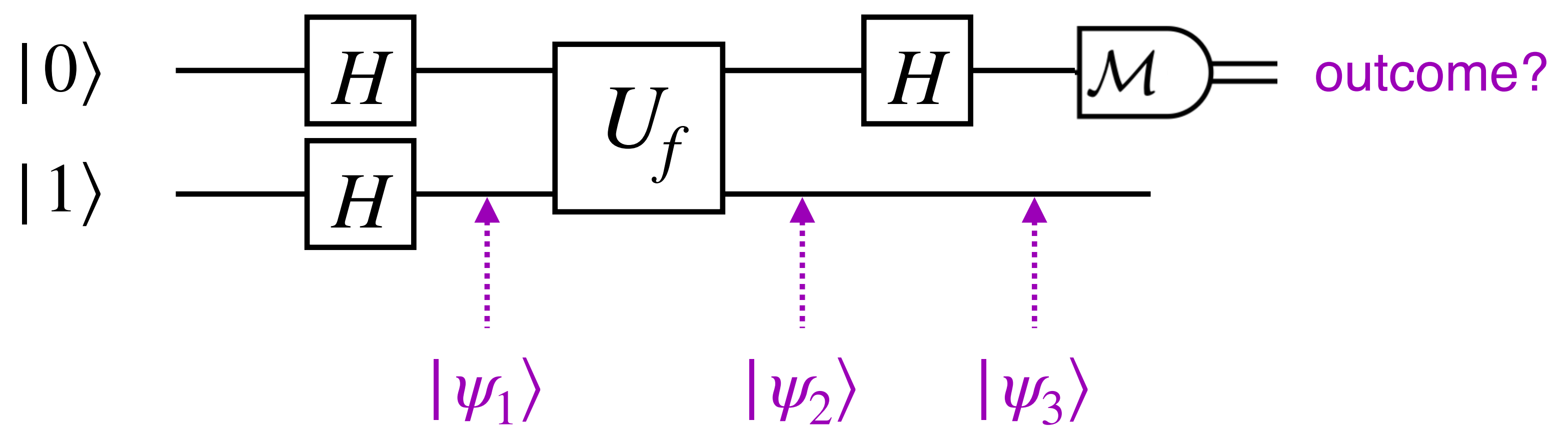
$$|0\rangle \overset{H}{\leftrightarrow} (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|1\rangle \overset{H}{\leftrightarrow} (|0\rangle - |1\rangle)/\sqrt{2}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \overset{H}{\leftrightarrow} (|0\rangle + |1\rangle)/\sqrt{2}$$

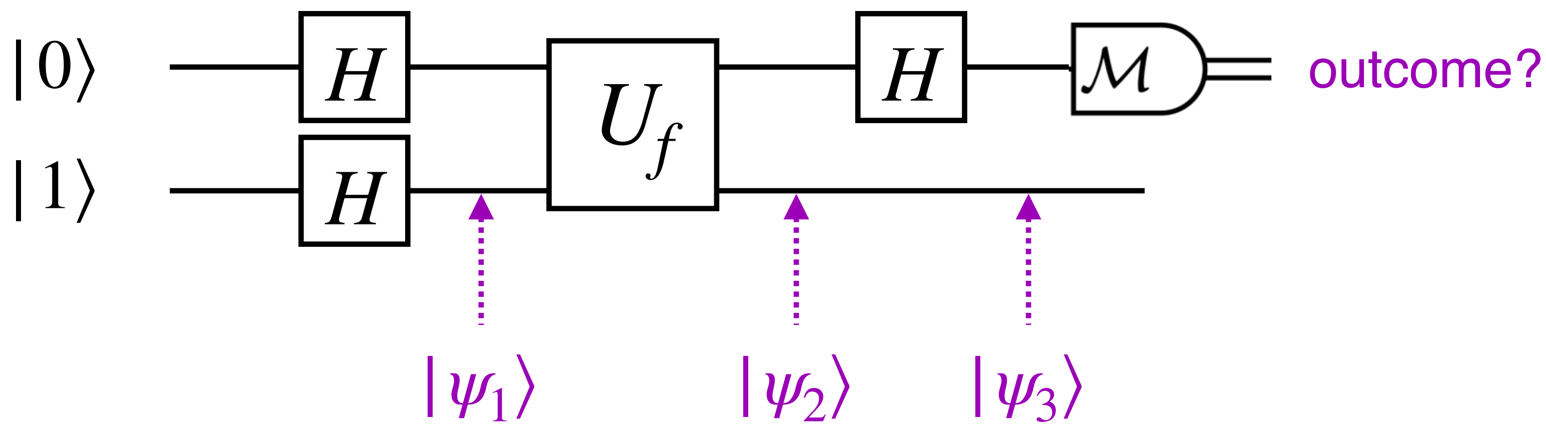
$$|1\rangle \overset{H}{\leftrightarrow} (|0\rangle - |1\rangle)/\sqrt{2}$$



$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \overset{H}{\leftrightarrow} (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|1\rangle \overset{H}{\leftrightarrow} (|0\rangle - |1\rangle)/\sqrt{2}$$

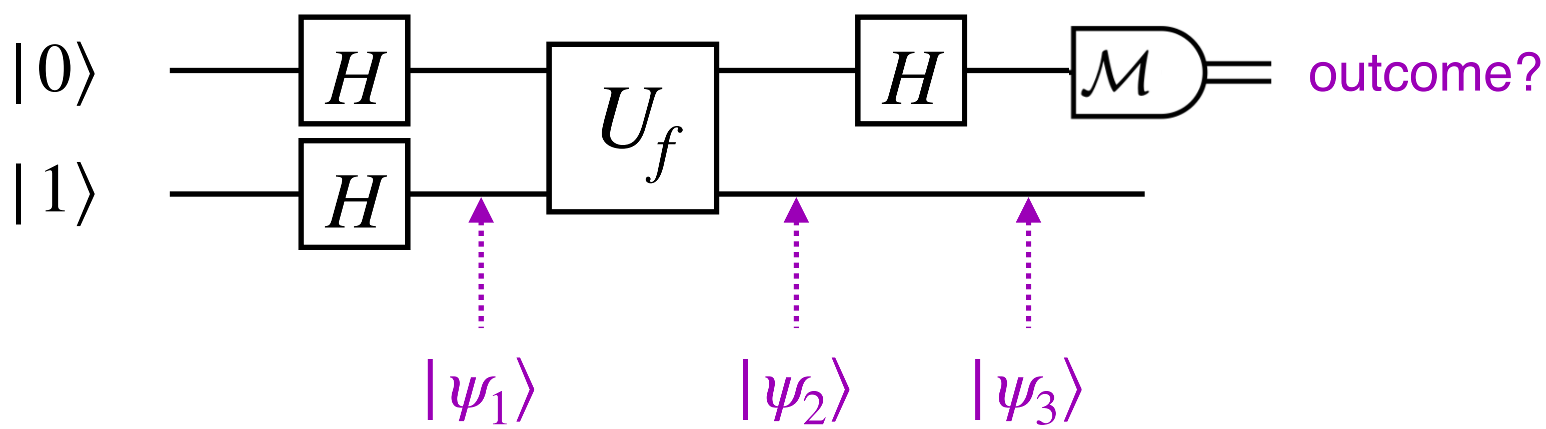


$$|\psi_1\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)/2 = (|0,0\rangle - |0,1\rangle + |1,0\rangle - |1,1\rangle)/2$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \overset{H}{\leftrightarrow} (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|1\rangle \overset{H}{\leftrightarrow} (|0\rangle - |1\rangle)/\sqrt{2}$$



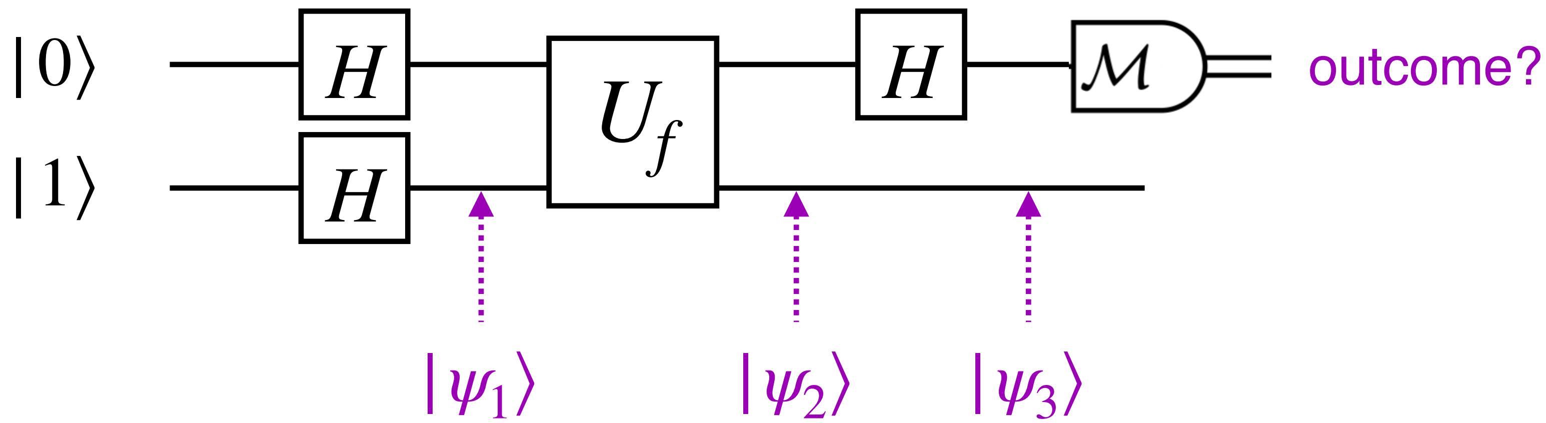
$$|\psi_1\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)/2 = (|0,0\rangle - |0,1\rangle + |1,0\rangle - |1,1\rangle)/2$$

$$|\psi_2\rangle = (|0,\ f(0)\rangle - |0,\ 1 \oplus f(0)\rangle + |1,\ f(1)\rangle - |1,\ 1 \oplus f(1)\rangle)/2$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \overset{H}{\leftrightarrow} (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|1\rangle \overset{H}{\leftrightarrow} (|0\rangle - |1\rangle)/\sqrt{2}$$



$$|\psi_1\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)/2 = (|0{,}0\rangle - |0{,}1\rangle + |1{,}0\rangle - |1{,}1\rangle)/2$$

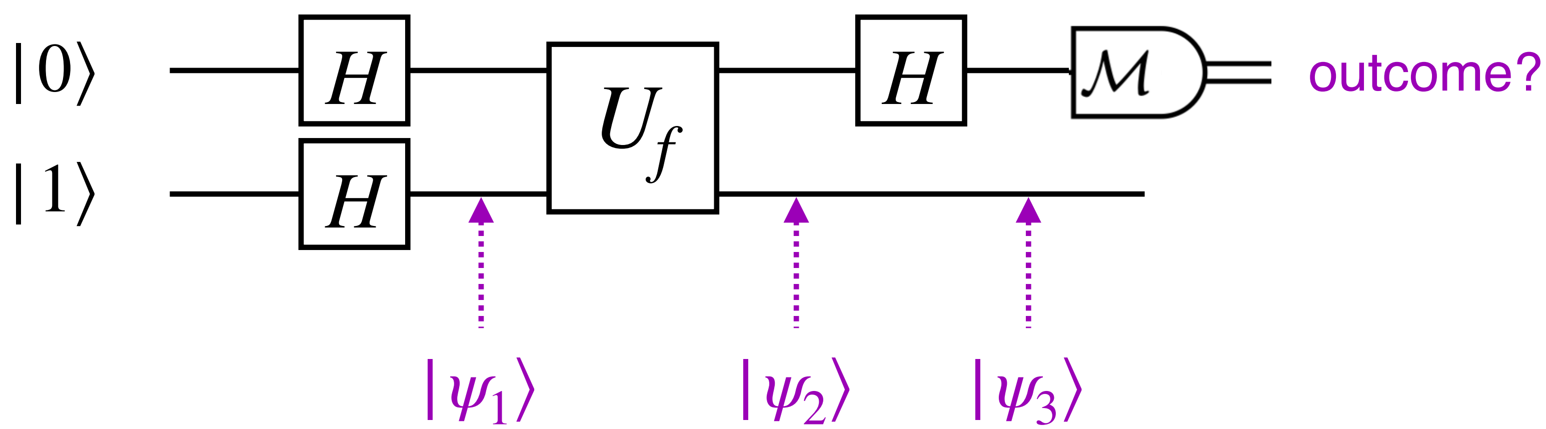
$$|\psi_2\rangle = (|0,\ f(0)\rangle - |0,\ 1 \oplus f(0)\rangle + |1,\ f(1)\rangle - |1,\ 1 \oplus f(1)\rangle)/2$$

$$= (-1)^{f(0)}|0\rangle \otimes (|0\rangle - |1\rangle)/2 + (-1)^{f(1)}|1\rangle \otimes (|0\rangle - |1\rangle)/2$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \overset{H}{\leftrightarrow} (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|1\rangle \overset{H}{\leftrightarrow} (|0\rangle - |1\rangle)/\sqrt{2}$$



$$|\psi_1\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)/2 = (|0{,}0\rangle - |0{,}1\rangle + |1{,}0\rangle - |1{,}1\rangle)/2$$

$$|\psi_2\rangle = (|0,\ f(0)\rangle - |0,\ 1 \oplus f(0)\rangle + |1,\ f(1)\rangle - |1,\ 1 \oplus f(1)\rangle)/2$$

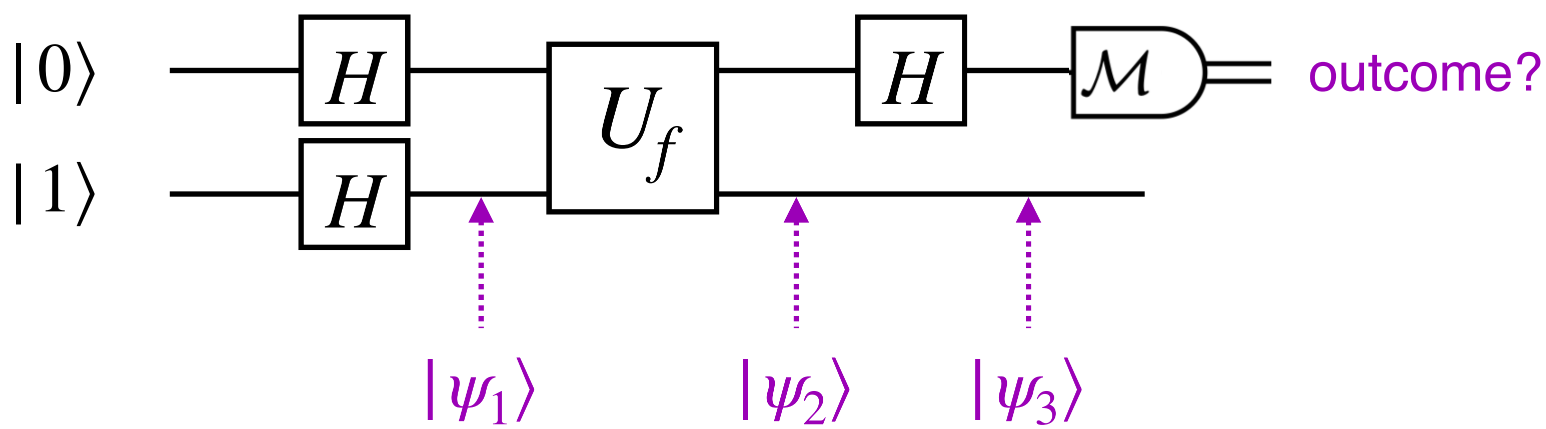
$$= (-1)^{f(0)}|0\rangle \otimes (|0\rangle - |1\rangle)/2 + (-1)^{f(1)}|1\rangle \otimes (|0\rangle - |1\rangle)/2$$

$$= (-1)^{f(0)} \frac{|0\rangle + (-1)^{f(1)-f(0)}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \overset{H}{\leftrightarrow} (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|1\rangle \overset{H}{\leftrightarrow} (|0\rangle - |1\rangle)/\sqrt{2}$$



$$|\psi_1\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)/2 = (|0,0\rangle - |0,1\rangle + |1,0\rangle - |1,1\rangle)/2$$

$$|\psi_2\rangle = (|0,\ f(0)\rangle - |0,\ 1 \oplus f(0)\rangle + |1,\ f(1)\rangle - |1,\ 1 \oplus f(1)\rangle)/2$$

$$= (-1)^{f(0)}|0\rangle \otimes (|0\rangle - |1\rangle)/2 + (-1)^{f(1)}|1\rangle \otimes (|0\rangle - |1\rangle)/2$$

$$= (-1)^{f(0)}\frac{|0\rangle + (-1)^{f(1)-f(0)}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \begin{cases} |0\rangle \otimes (|0\rangle - |1\rangle)/\sqrt{2} & \text{f constant} \\ |1\rangle \otimes (|0\rangle - |1\rangle)/\sqrt{2} & \text{f balanced} \end{cases}$$

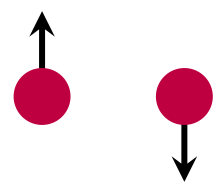
outcome = 0 $\Longrightarrow$ *f* constant

outcome = 1 $\Longrightarrow$ *f* balanced

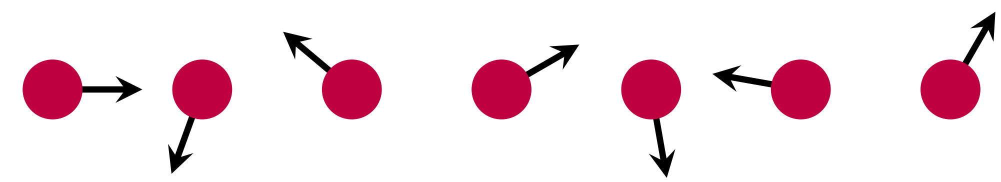
Part I: Outline

- Basics of quantum computing/mechanics
- Motivations to topological quantum computing (TQC) and what it is
- Anyons and topological phases of matter from lattice models

Qubits: abstraction of quantum systems, e.g. spin

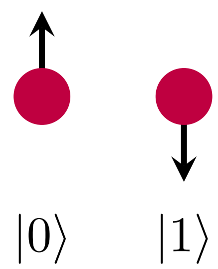


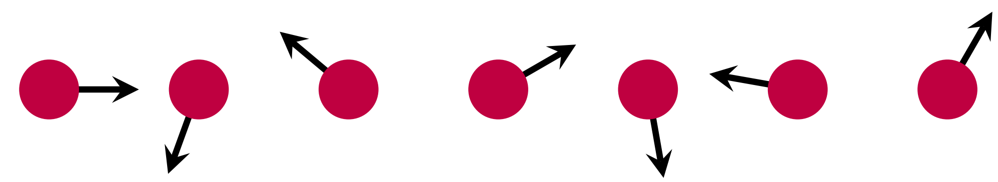
$|0\rangle$ $|1\rangle$



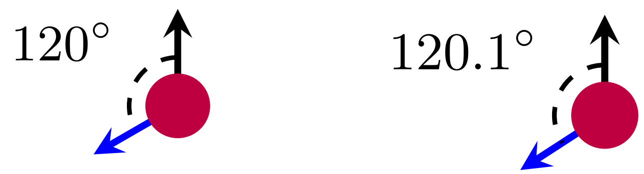
multi-qubit state

Qubits: abstraction of quantum systems, e.g. spin

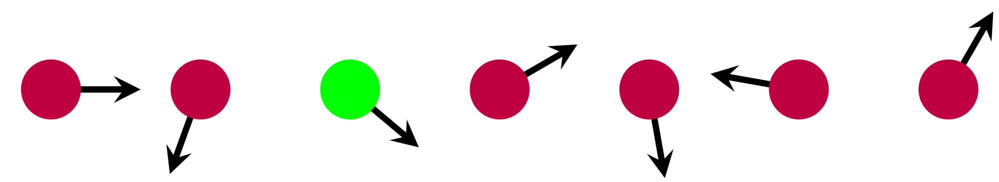




multi-qubit state

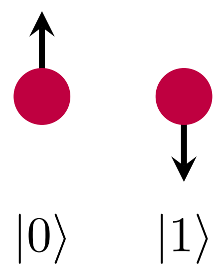


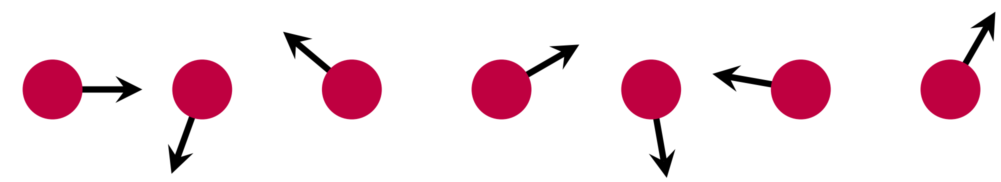
imperfections



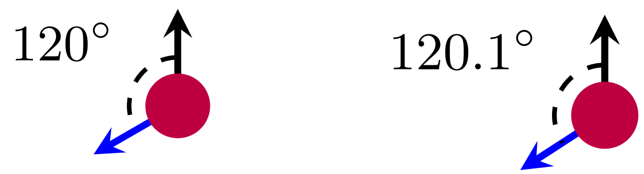
environment interaction

Qubits: abstraction of quantum systems, e.g. spin

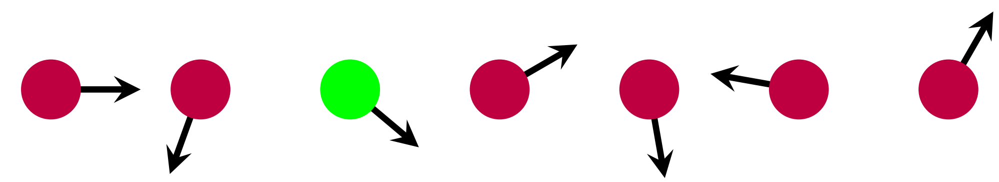




multi-qubit state

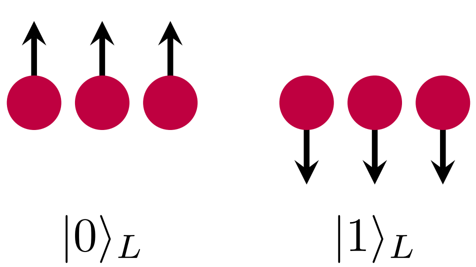


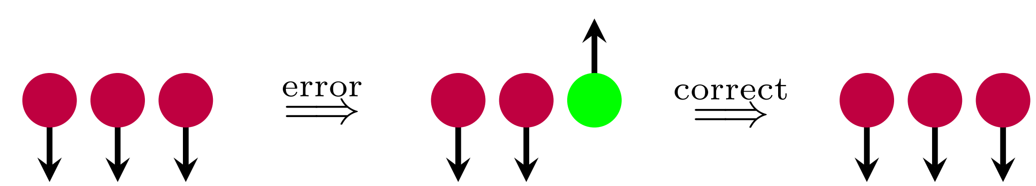
imperfections



environment interaction

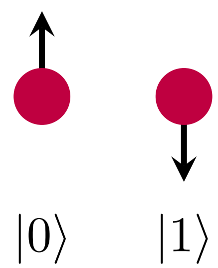
★ ‘Software’ approach: quantum error correction

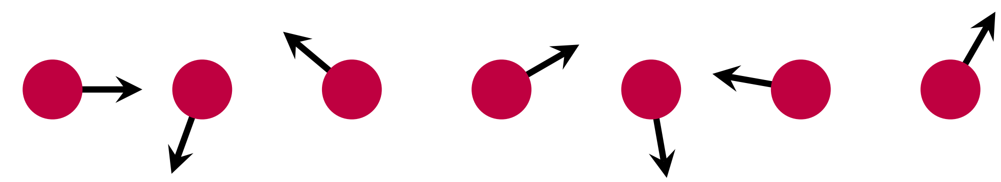




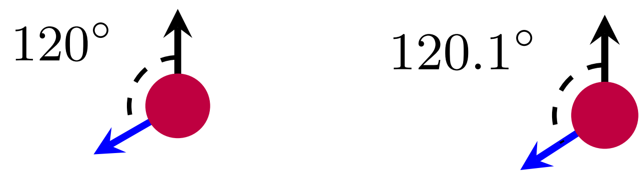
More sophisticated error correction schemes exist!

Qubits: abstraction of quantum systems, e.g. spin

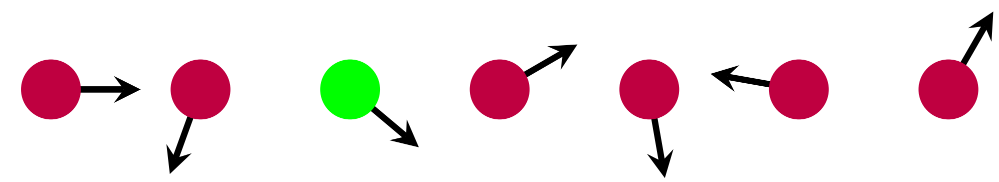




multi-qubit state

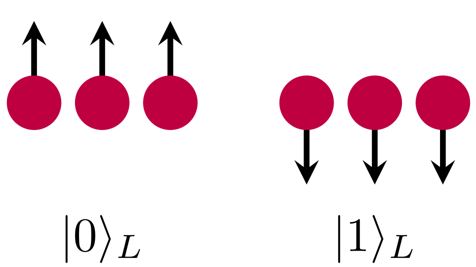


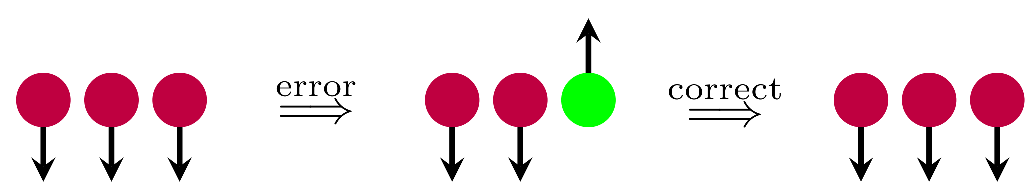
imperfections



environment interaction

★ ‘Software’ approach: quantum error correction





More sophisticated error correction schemes exist!

★ ‘Hardware’ approach: topological quantum computation (TQC)

Topological idea:

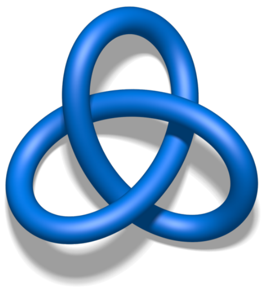
Encode information in global properties of a quantum system, so that no small local perturbation can easily change the encoded state

Topological idea:

Encode information in global properties of a quantum system, so that no small local perturbation can easily change the encoded state

Analogy:

local properties:
length, curvature, etc



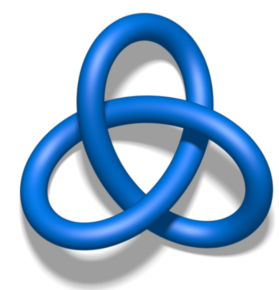
global property:
knot-type

Topological idea:

Encode information in global properties of a quantum system, so that no small local perturbation can easily change the encoded state

Analogy:

local properties:
length, curvature, etc



global property:
knot-type

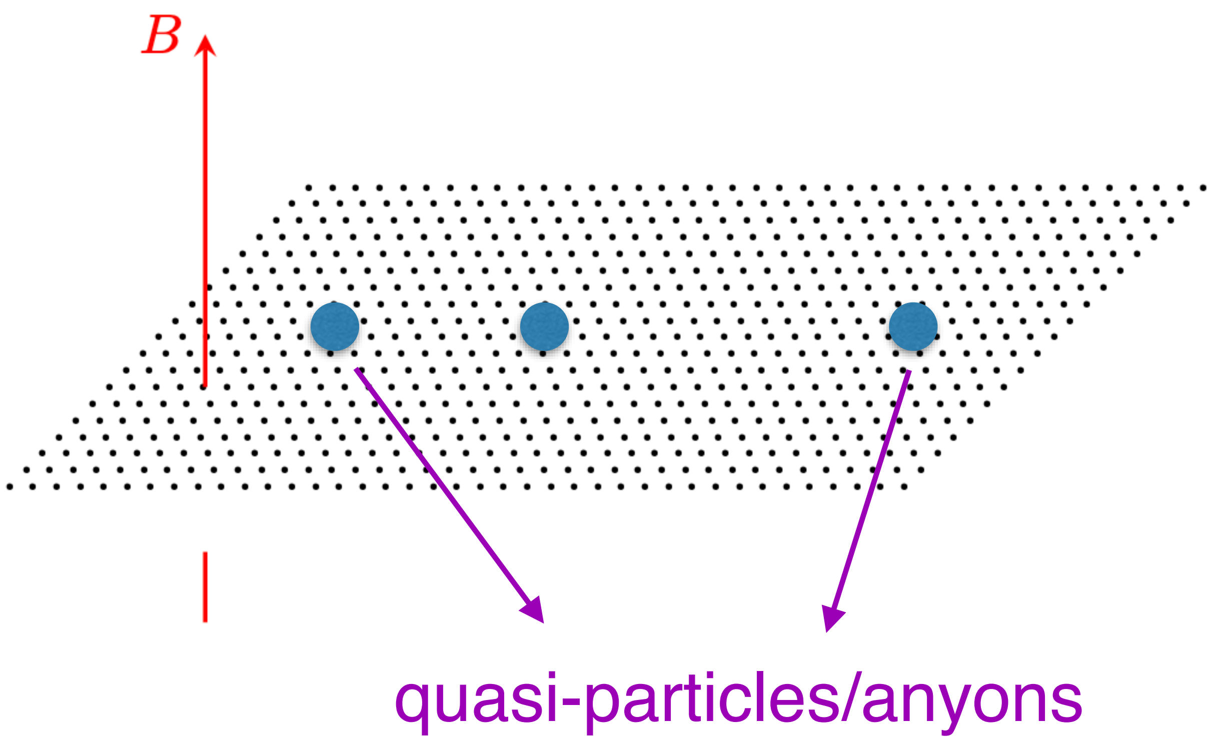
- Store quantum information in global/topological sectors;
- Perform computation by changing the global/topological configuration in controlled ways;

A class of quantum systems with the above properties are called topological phases of matter

Topological phases of matter in 2D

Electrons constrained in a plane
with a strong magnetic field,
at very low temperature.

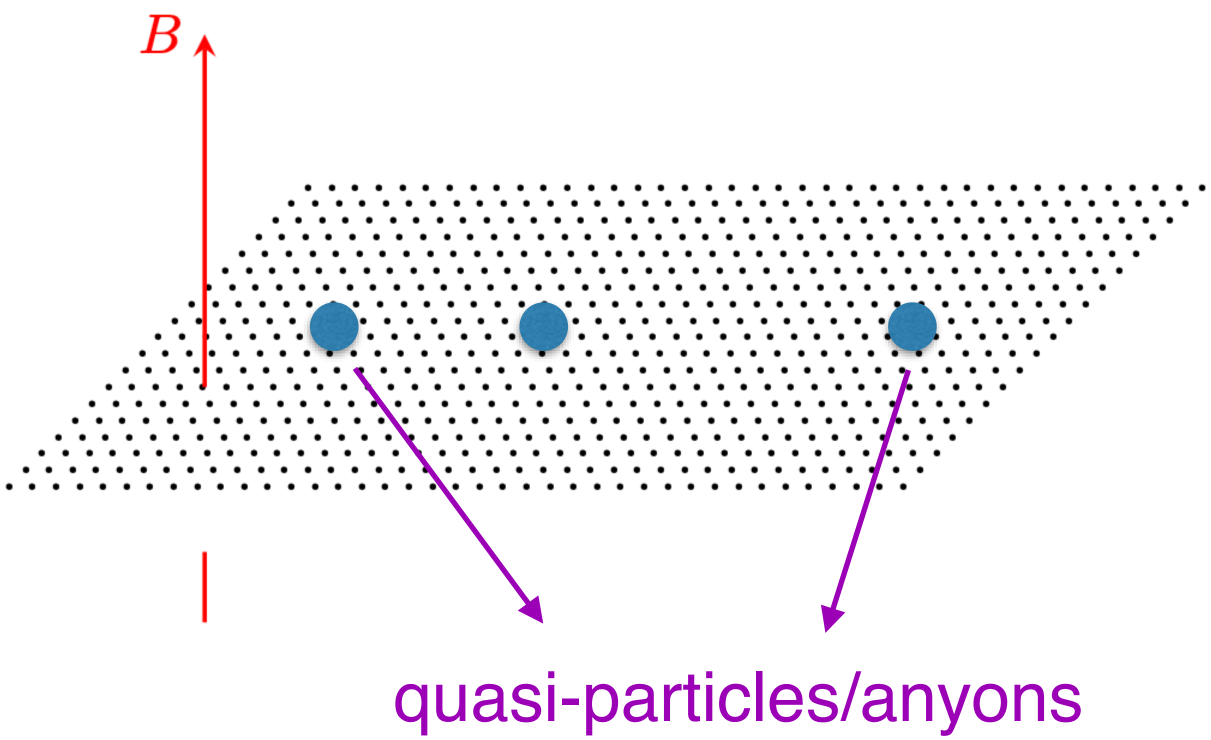
Topological phases of matter in 2D



Electrons constrained in a plane
with a strong magnetic field,
at very low temperature.

- Quasi-particles: collective behavior of the underlying particles.

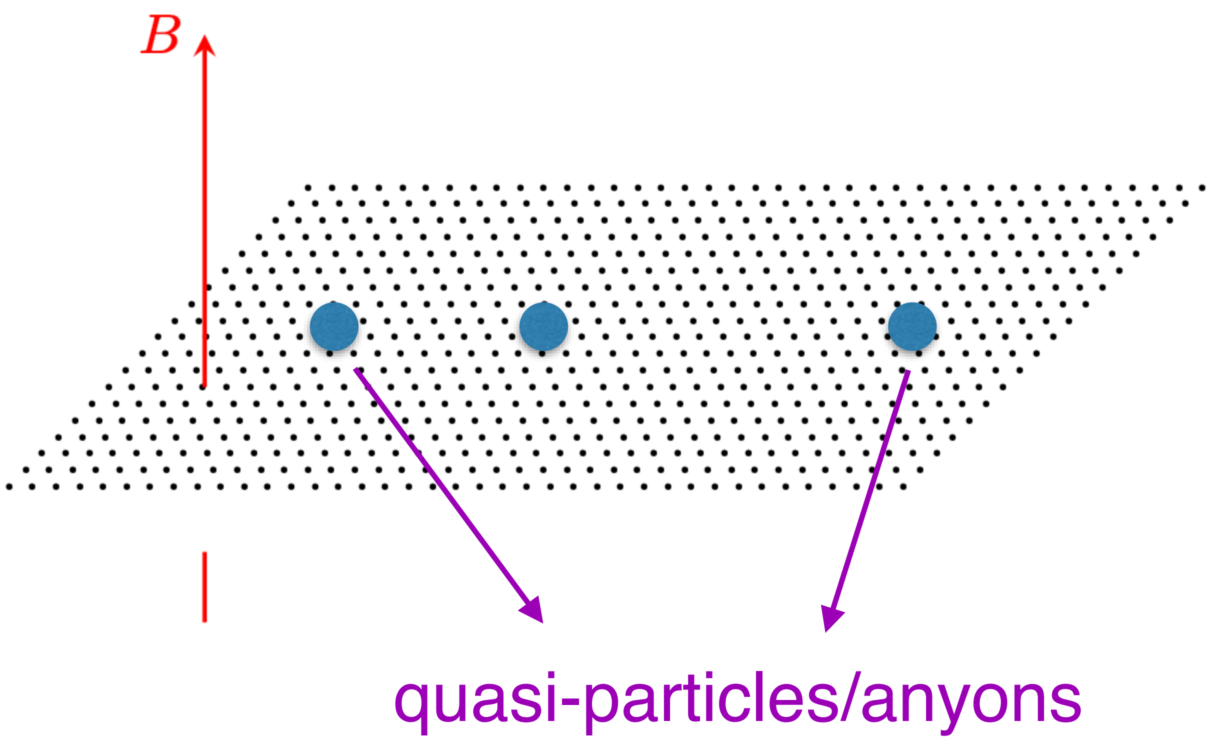
Topological phases of matter in 2D



Electrons constrained in a plane
with a strong magnetic field,
at very low temperature.

- Quasi-particles: collective behavior of the underlying particles.
- They can only be created and annihilated in pairs.

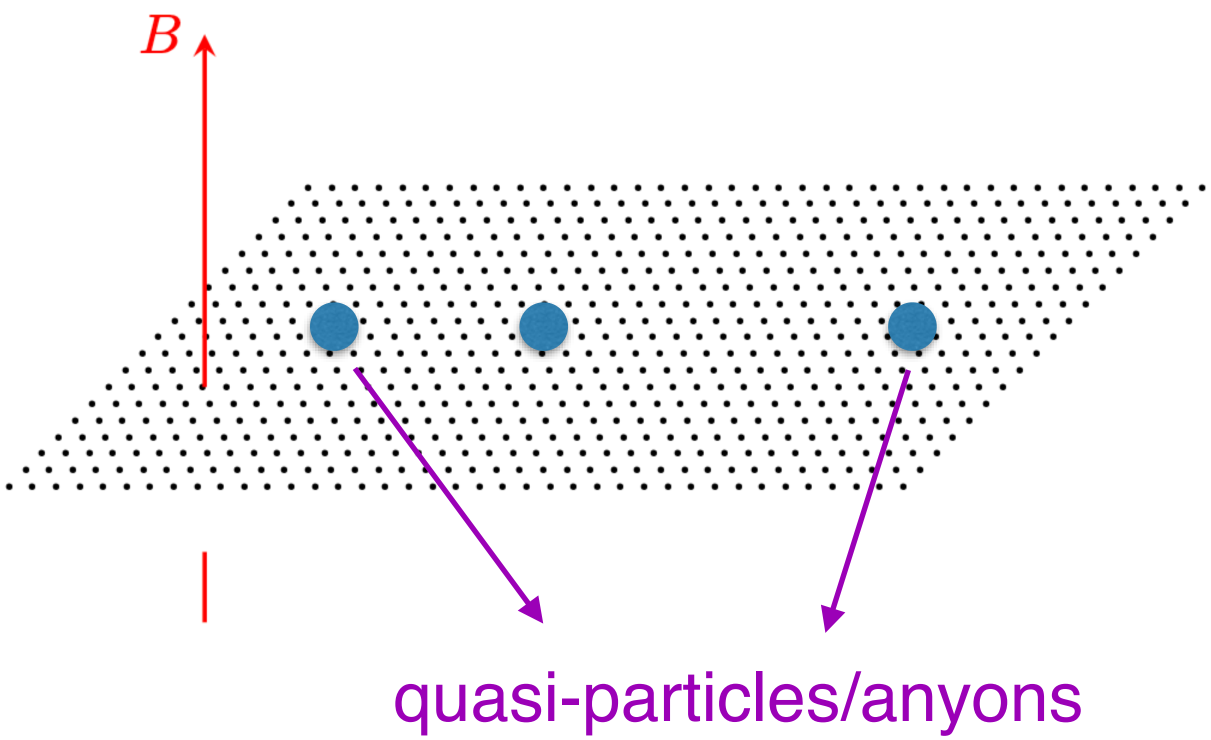
Topological phases of matter in 2D



Electrons constrained in a plane
with a strong magnetic field,
at very low temperature.

- Quasi-particles: collective behavior of the underlying particles.
- They can only be created and annihilated in pairs.
- The total type of two separated quasi-particles cannot be inferred locally.

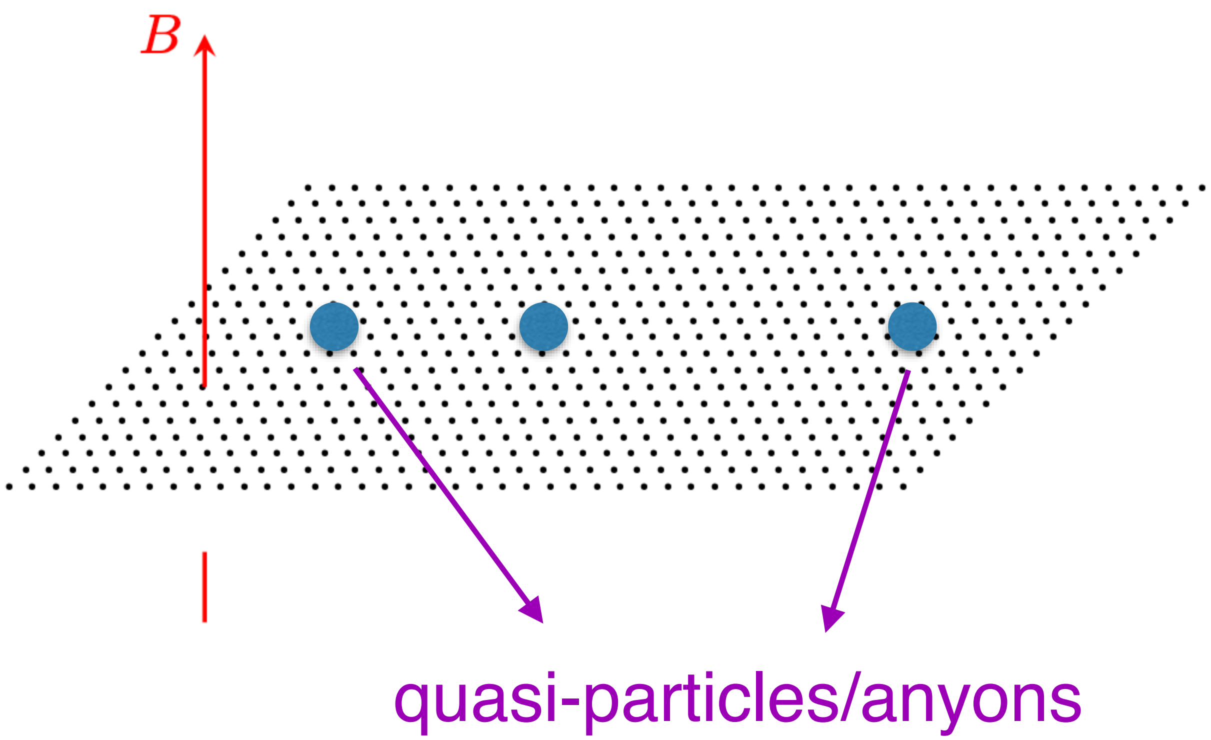
Topological phases of matter in 2D



Electrons constrained in a plane
with a strong magnetic field,
at very low temperature.

- Quasi-particles: collective behavior of the underlying particles.
- They can only be created and annihilated in pairs.
- The total type of two separated quasi-particles cannot be inferred locally.
- The state space of quasi-particles depends on their types and background topology.

Topological phases of matter in 2D

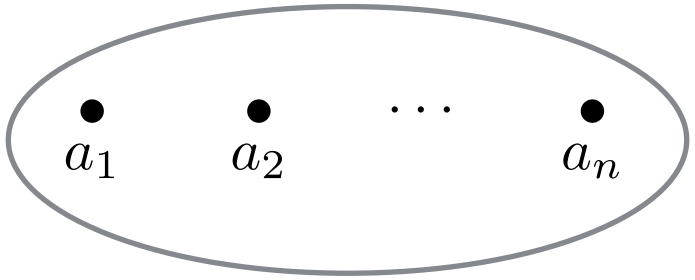


Electrons constrained in a plane
with a strong magnetic field,
at very low temperature.

- Quasi-particles: collective behavior of the underlying particles.
- They can only be created and annihilated in pairs.
- The total type of two separated quasi-particles cannot be inferred locally.
- The state space of quasi-particles depends on their types and background topology.
- Moving the quasi-particles induce robust transformation on state spaces.

★ Qubit: global degrees of freedom of multiple-anyon space

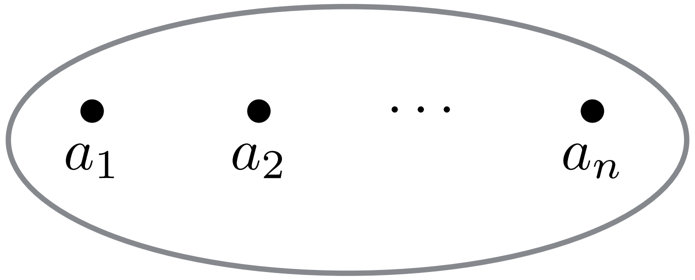
e.g. the fusion channel



- The space is inaccessible by local small perturbations.
- There is a gap with the rest of the spectrum.

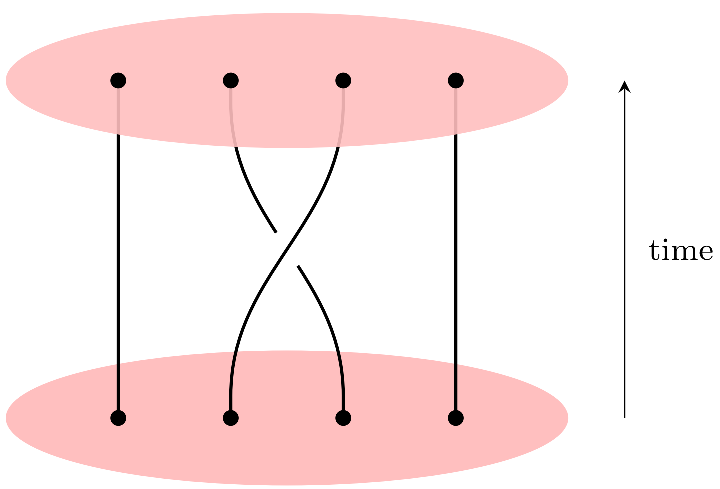
★ Qubit: global degrees of freedom of multiple-anyon space

e.g. the fusion channel



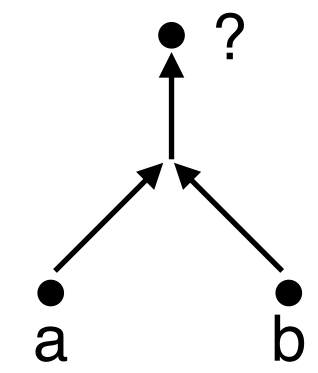
- The space is inaccessible by local small perturbations.
- There is a gap with the rest of the spectrum.

★ Quantum gates: braiding/swap of anyons

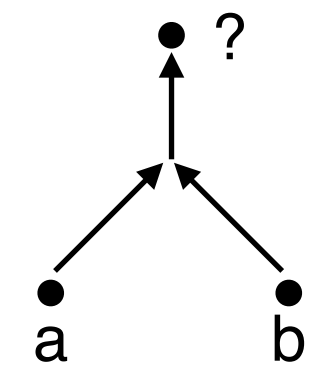


- Braiding induces unitary transformation;
- The transformation depends only on the isotopy class of trajectories.

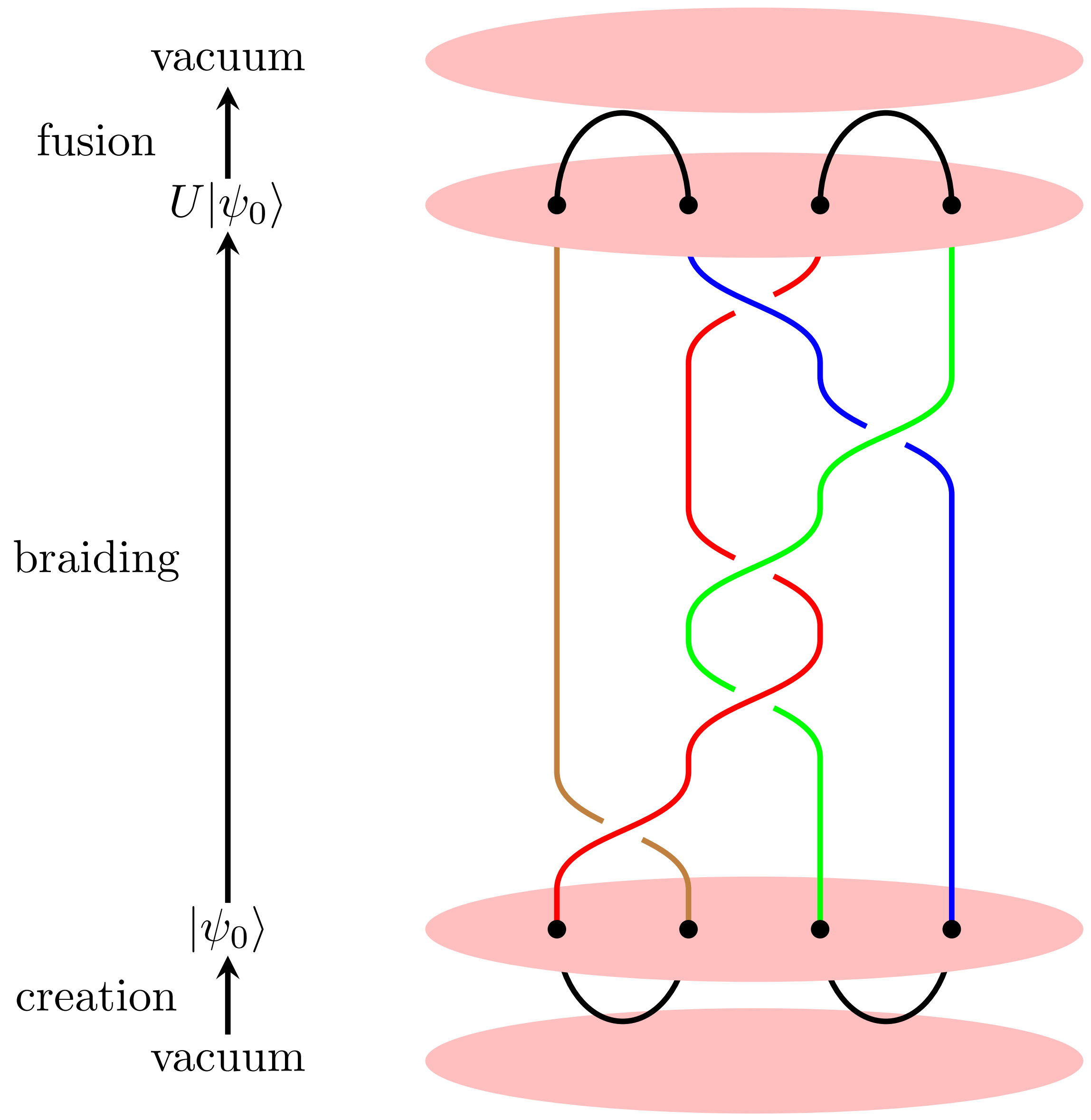
★ Measurement: anyon fusion



★ Measurement: anyon fusion



Process of topological quantum computing



A. Yu. Kitaev, Fault-tolerant quantum computation by anyons, arXiv:quant-ph/9707021,

M.H. Freedman, P/NP, and the quantum field computer, Proc. Natl. Acad. Sci. 95 (1) 98-101,

M.H. Freedman, A. Yu. Kitaev, M.J. Larsen, Z. Wang, Topological Quantum Computation, arXiv:quant-ph/0101025,

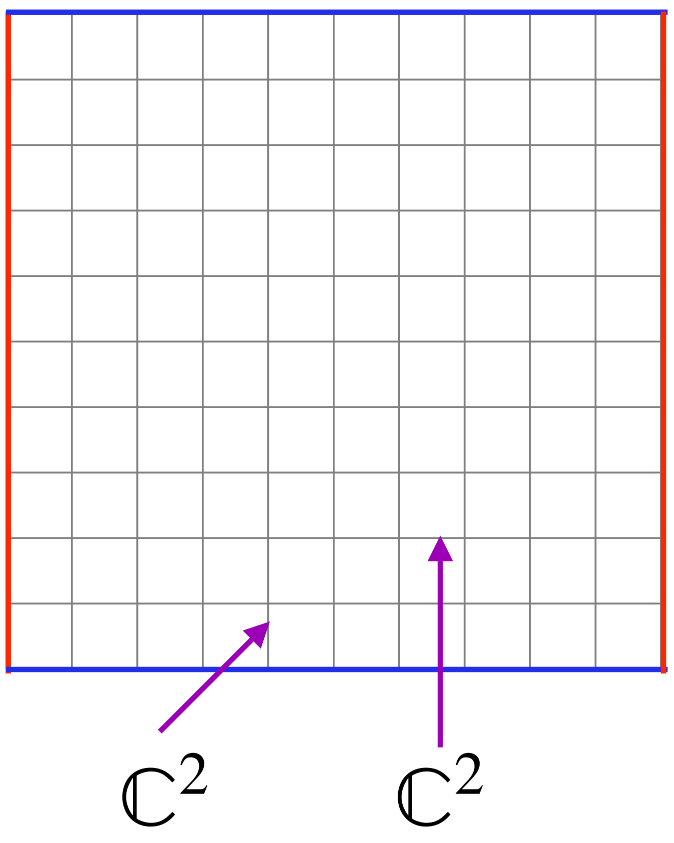
Part I: Outline

- Basics of quantum computing/mechanics
- Motivations to topological quantum computing (TQC) and what it is
- Anyons and topological phases of matter from lattice models

Toric code model

Choose a closed surface Σ and a cellullation/lattice with *(V, E, F)*

vertices edges faces

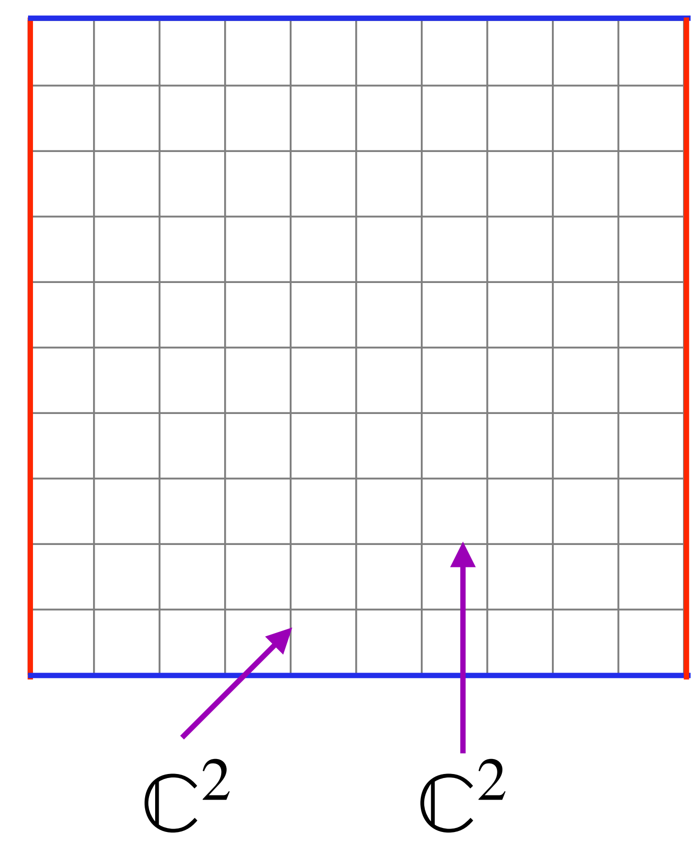


Toric code model

Choose a closed surface Σ and a cellullation/lattice with *(V, E, F)*

vertices edges faces

Assign a qubit $\mathbb{C}^2 = \{\,|0\rangle, |1\rangle\}$ on each edge



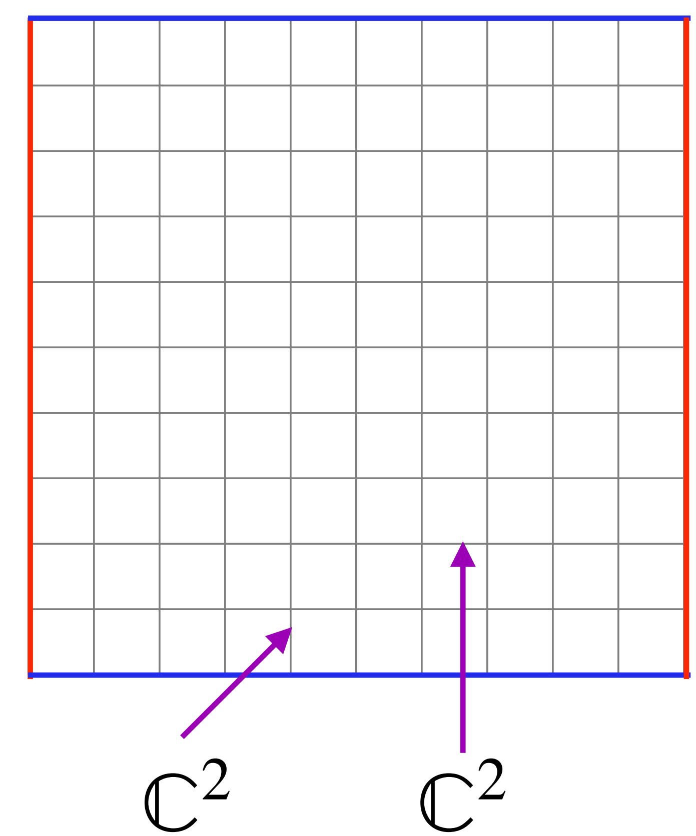
Toric code model

Choose a closed surface Σ and a cellullation/lattice with *(V, E, F)*

vertices edges faces

Assign a qubit $\mathbb{C}^2 = \{\,|0\rangle, |1\rangle\}$ on each edge

Total Hilbert/state space $\mathscr{H} = \bigotimes_{e \in E} (\mathbb{C}^2)_e$

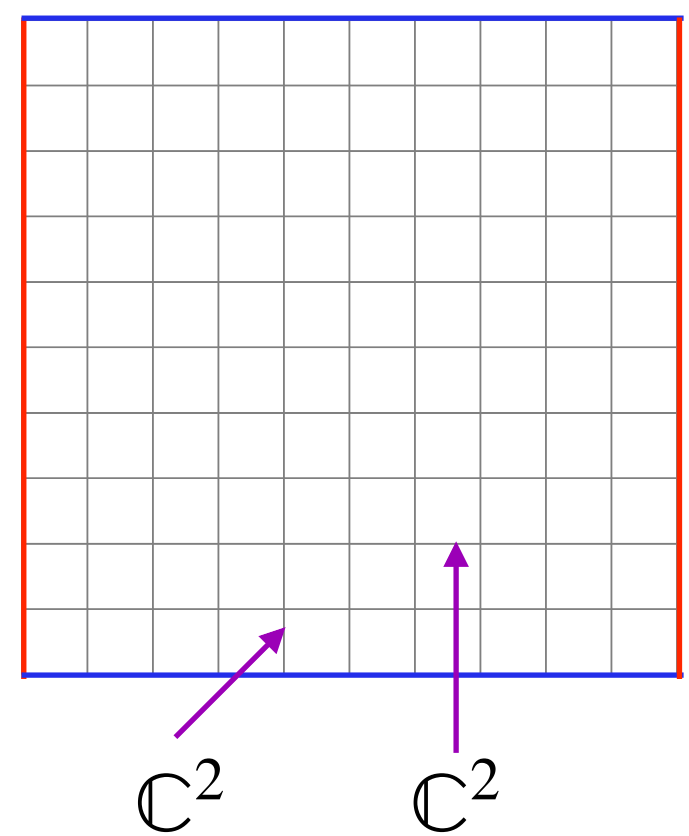


Toric code model

Choose a closed surface Σ and a cellullation/lattice with *(V, E, F)*

vertices edges faces

Assign a qubit $\mathbb{C}^2 = \{ |0\rangle, |1\rangle \}$ on each edge



Total Hilbert/state space $\mathscr{H} = \otimes_{e \in E} (\mathbb{C}^2)_e$

A standard basis of $\mathscr{H}$ consists of
$|s\rangle := \otimes_{e \in E} |s(e)\rangle$, for each $s : E \to \mathbb{Z}_2 = \{0,1\}$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X = X^\dagger$, $Z = Z^\dagger$ $\qquad X^2 = Z^2 = I$

$XZ = -ZX$

The eigenvalues of X and Z are ± 1

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X = X^\dagger$, $Z = Z^\dagger$ $\qquad$ $X^2 = Z^2 = I$

$XZ = -ZX$

The eigenvalues of X and Z are ± 1

Remark: if O acts on $\mathbb{C}^2$, $e \in E$, then O_e acts on the copy $(\mathbb{C}^2)_e$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X = X^\dagger$, $Z = Z^\dagger$ $\qquad X^2 = Z^2 = I$

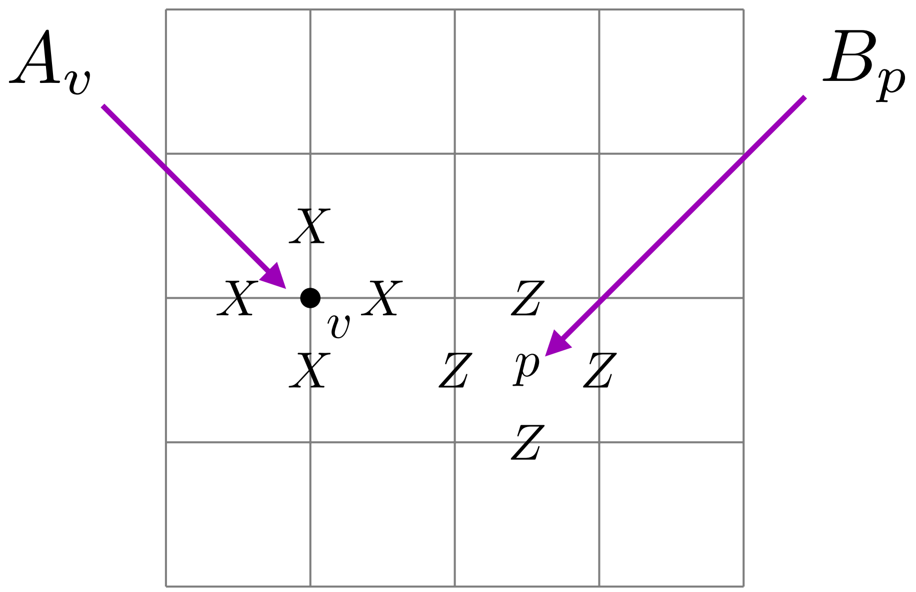
$XZ = -ZX$

The eigenvalues of X and Z are ± 1

Remark: if O acts on $\mathbb{C}^2$, $e \in E$, then O_e acts on the copy $(\mathbb{C}^2)_e$.

For $v \in V$, $p \in F$, define operators:

$$A_v := \bigotimes_{e:v\in\partial e} X_e \bigotimes_{e:v\notin\partial e} I_e \qquad B_p := \bigotimes_{e:e\in\partial p} Z_e \bigotimes_{e:e\notin\partial p} I_e$$



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X = X^\dagger$, $Z = Z^\dagger$ $\qquad X^2 = Z^2 = I$

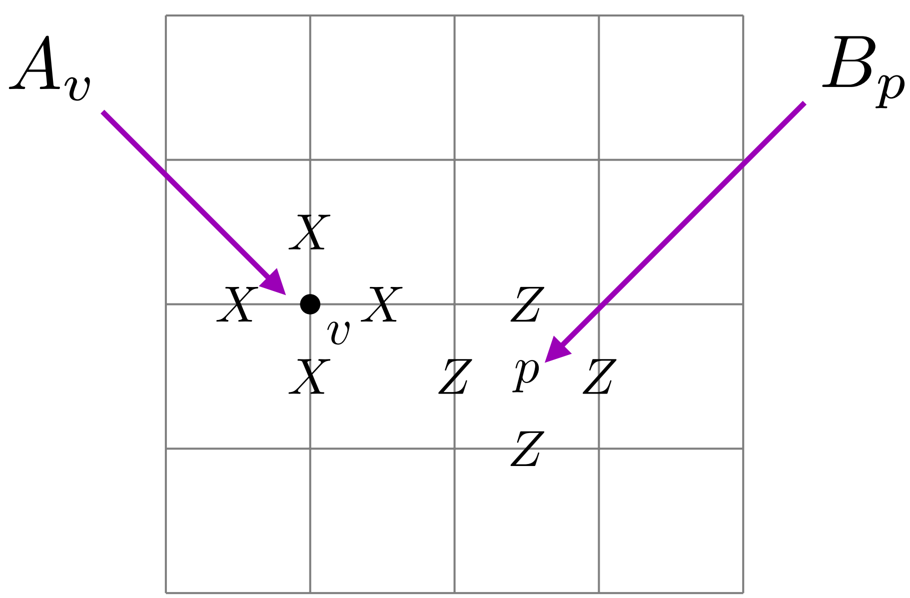
$XZ = -ZX$

The eigenvalues of X and Z are ± 1

Remark: if O acts on $\mathbb{C}^2$, $e \in E$, then O_e acts on the copy $(\mathbb{C}^2)_e$.

For $v \in V$, $p \in F$, define operators:

$$A_v := \bigotimes_{e: v \in \partial e} X_e \bigotimes_{e: v \notin \partial e} I_e \qquad B_p := \bigotimes_{e: e \in \partial p} Z_e \bigotimes_{e: e \notin \partial p} I_e$$



- $A_v = A_v^\dagger$, $\; B_p = B_p^\dagger$
- $A_v^2 = B_p^2 = I$
- $\{A_v, B_p \mid v \in V, p \in F\}$ pairwise commute

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X = X^\dagger$, $Z = Z^\dagger$ $\qquad X^2 = Z^2 = I$

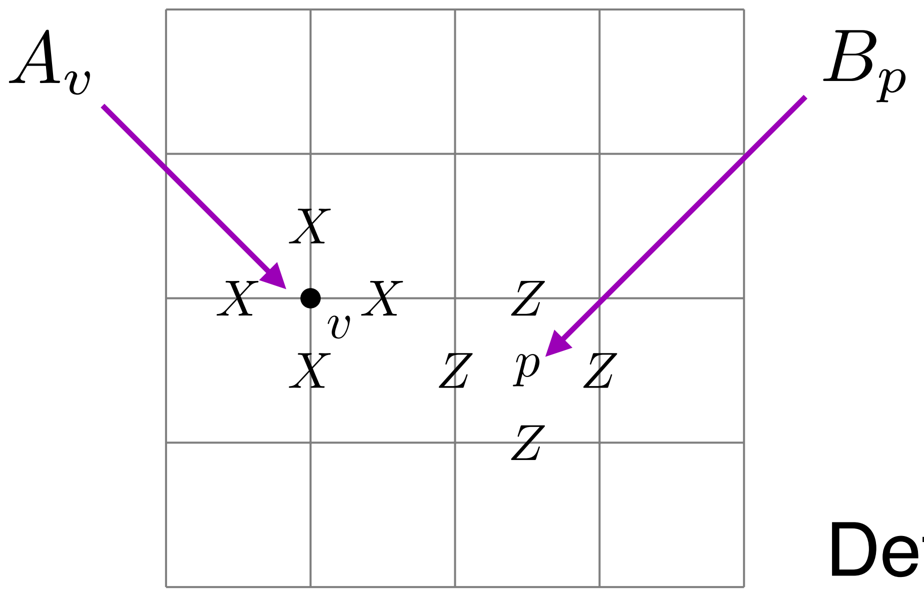
$XZ = -ZX$

The eigenvalues of X and Z are ± 1

Remark: if O acts on $\mathbb{C}^2$, $e \in E$, then O_e acts on the copy $(\mathbb{C}^2)_e$.

For $v \in V$, $p \in F$, define operators:

$$A_v := \bigotimes_{e: v \in \partial e} X_e \bigotimes_{e: v \notin \partial e} I_e \qquad B_p := \bigotimes_{e: e \in \partial p} Z_e \bigotimes_{e: e \notin \partial p} I_e$$



- $A_v = A_v^\dagger$, $\;B_p = B_p^\dagger$
- $A_v^2 = B_p^2 = I$
- $\{A_v, B_p \mid v \in V, p \in F\}$ pairwise commute

Define the Hamiltonian $\quad H = -\sum_{v \in V} A_v - \sum_{p \in F} B_p$

★The ground state space

i.e. the eigenspace of *H* with lowest eigenvalue

$$V_{gs} = \{|\psi\rangle \in \mathcal{H} : \ A_v|\psi\rangle = B_p|\psi\rangle = |\psi\rangle, \ \forall v \in V, p \in F\}$$

★The ground state space

i.e. the eigenspace of *H* with lowest eigenvalue

$$V_{gs} = \{|\psi\rangle \in \mathcal{H} : \; A_v|\psi\rangle = B_p|\psi\rangle = |\psi\rangle, \; \forall v \in V, p \in F\}$$

$$X|0\rangle = |1\rangle \qquad Z|0\rangle = |0\rangle$$

$$X|1\rangle = |0\rangle \qquad Z|1\rangle = -|1\rangle$$

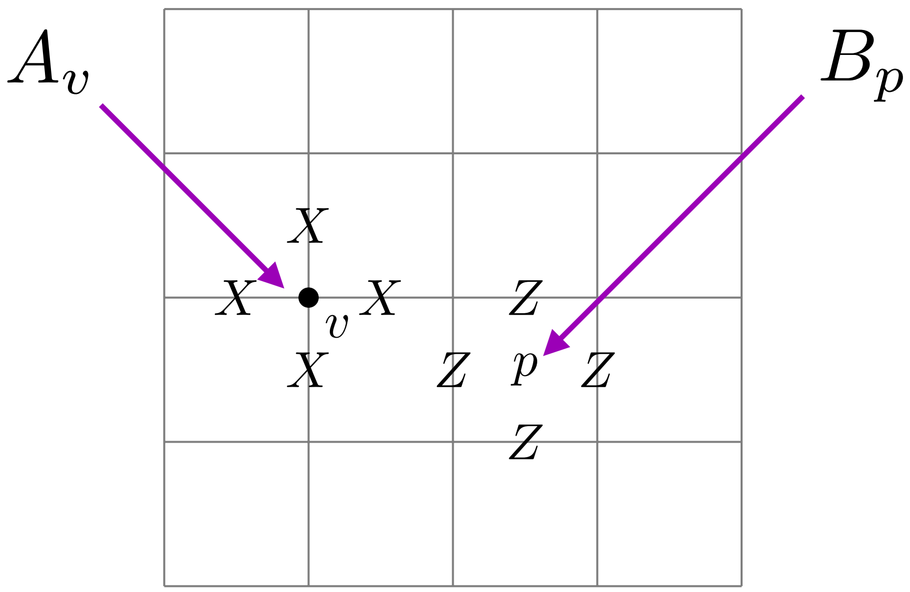
★The ground state space

i.e. the eigenspace of *H* with lowest eigenvalue

$$V_{gs} = \{|\psi\rangle \in \mathcal{H} : \ A_v|\psi\rangle = B_p|\psi\rangle = |\psi\rangle, \ \forall v \in V, p \in F\}$$

$$\boxed{\begin{array}{ll} X|0\rangle = |1\rangle & Z|0\rangle = |0\rangle \\ X|1\rangle = |0\rangle & Z|1\rangle = -|1\rangle \end{array}}$$

- B_p is diagonal under the standard basis. For $s : E \to \mathbb{Z}_2$



$$B_p|s\rangle = |s\rangle \iff \sum_{e \in \partial p} s(e) = 0 \in \mathbb{Z}_2$$

Consider V, E, F as 0-, 1-, 2-cells. Co-chain complex:

$$C^0 = \{g : V \to \mathbb{Z}_2\} \xrightarrow{\delta^0} C^1 = \{s : E \to \mathbb{Z}_2\} \xrightarrow{\delta^1} C^2 = \{f : F \to \mathbb{Z}_2\}$$

For $s \in C^1$, $\qquad B_p|s\rangle = |s\rangle, \forall p \in F \quad \Longleftrightarrow \quad \delta^1(s) = 0$

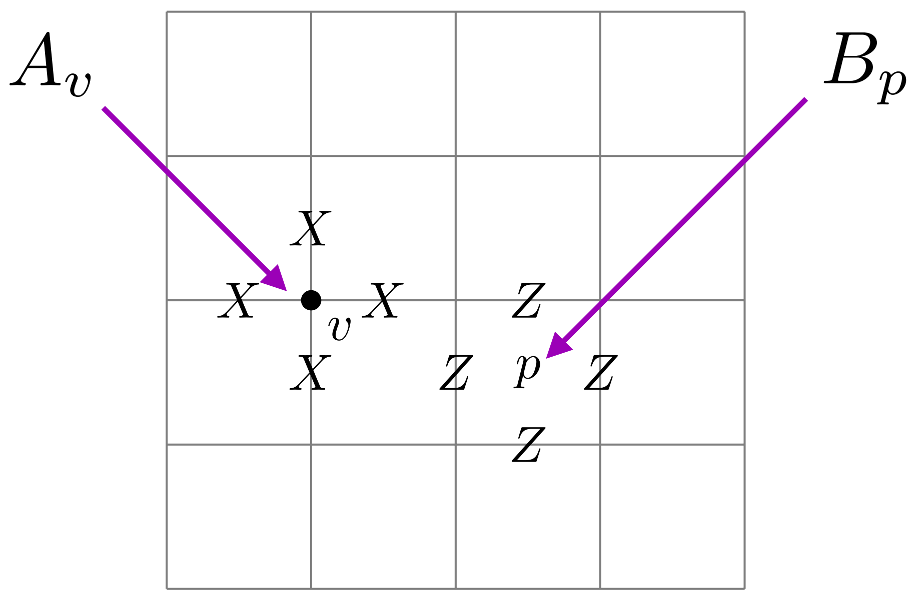
Consider V, E, F as 0-, 1-, 2-cells. Co-chain complex:

$$C^0 = \{g : V \to \mathbb{Z}_2\} \xrightarrow{\delta^0} C^1 = \{s : E \to \mathbb{Z}_2\} \xrightarrow{\delta^1} C^2 = \{f : F \to \mathbb{Z}_2\}$$

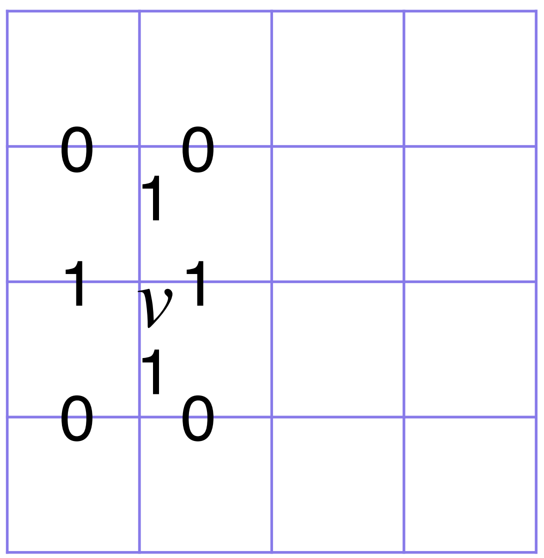
For $s \in C^1$, $\qquad B_p|s\rangle = |s\rangle, \forall p \in F \quad \Longleftrightarrow \quad \delta^1(s) = 0$

- A_v permutes elements of $\ker \delta^1$.

For $v \in V$, let $g_v : V \to \mathbb{Z}_2$, $g_v(u) = \delta_{u,v}$.



$\delta^0(g_v)$



$$A_v|s\rangle = |s + \delta^0(g_v)\rangle$$

- The dimension of V_{gs}.

Define $s \sim s'$ if $s - s' \in \operatorname{im} \delta^0$, and denote by $[s]$ the equivalence class.

- The dimension of V_{gs}.

Define $s \sim s'$ if $s - s' \in \operatorname{im} \delta^0$, and denote by $[s]$ the equivalence class.

➡ The action of the A_v's permutes elements within each $[s]$.

- The dimension of V_{gs}.

Define $s \sim s'$ if $s - s' \in \operatorname{im} \delta^0$, and denote by $[s]$ the equivalence class.

➡ The action of the A_v 's permutes elements within each $[s]$.

➡ Let $|[s]\rangle := \sum_{s' \in [s]} |s'\rangle$. Then $A_v |[s]\rangle = |[s]\rangle, \quad \forall v$.

- The dimension of V_{gs}.

Define $s \sim s'$ if $s - s' \in \mathrm{im}\,\delta^0$, and denote by $[s]$ the equivalence class.

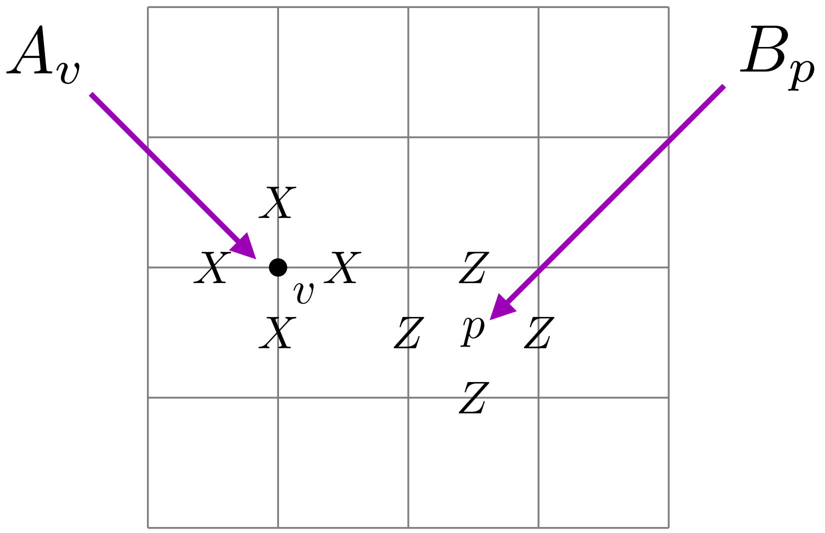
➡ The action of the A_v 's permutes elements within each $[s]$.

➡ Let $|[s]\rangle := \sum_{s' \in [s]} |s'\rangle$. Then $A_v|[s]\rangle = |[s]\rangle, \quad \forall v$.

Conclusion: $\dim V_{gs}$ equals the size of $H^1(\Sigma;\mathbb{Z}_2) := \ker \delta^1 / \mathrm{im}\,\delta^0$, independent of the lattice.

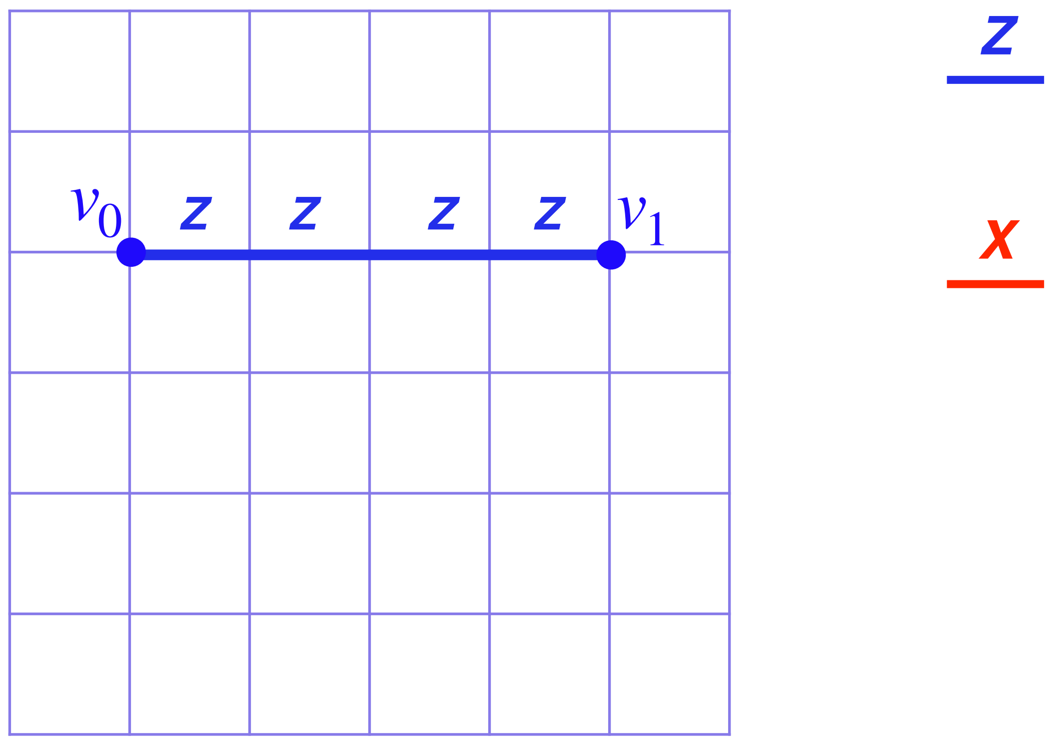
★Excited states:

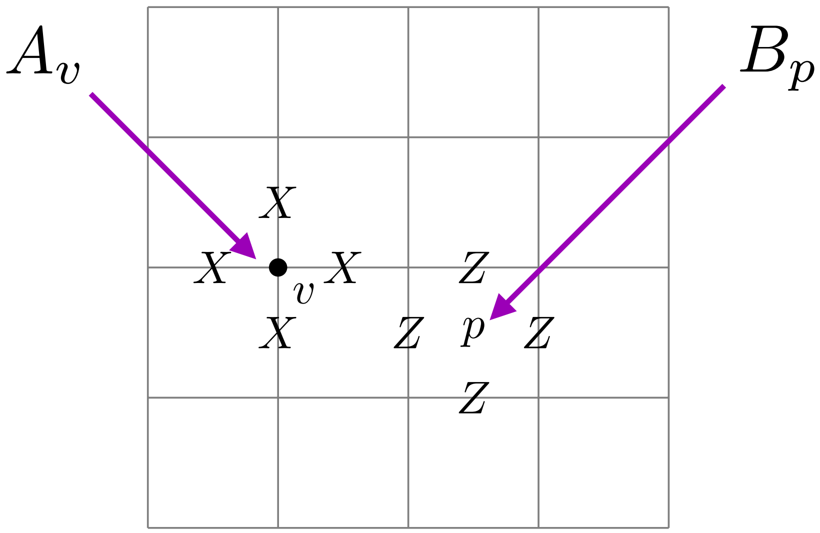
eigenstates of H besides ground states



★Excited states:

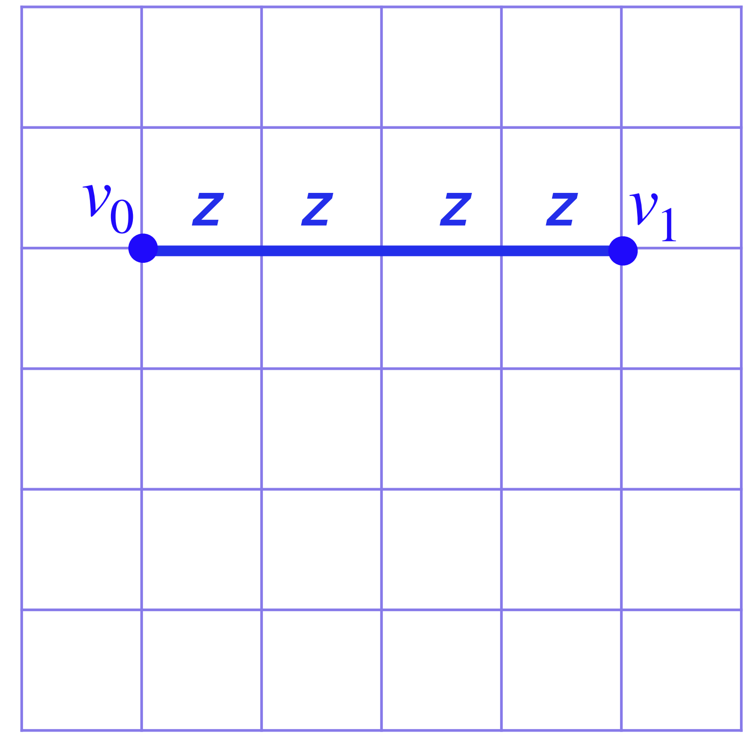
eigenstates of H besides ground states





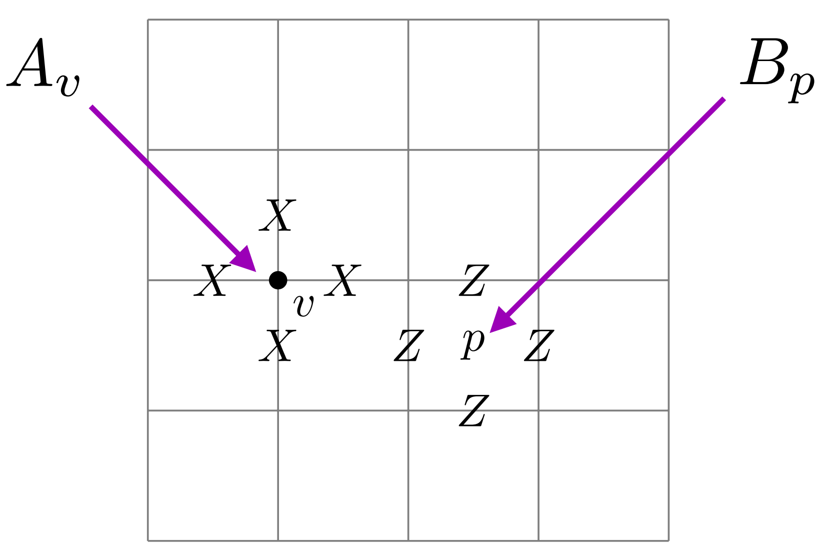
★Excited states:

eigenstates of H besides ground states



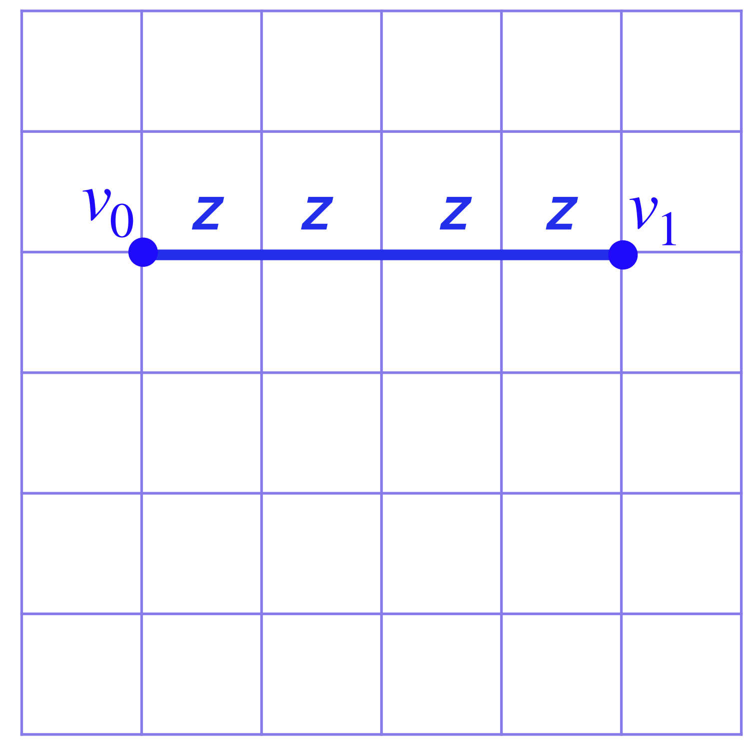
For a string *L* of the lattice,

$$Z_L := \bigotimes_{e \in L} Z_e$$



★Excited states:

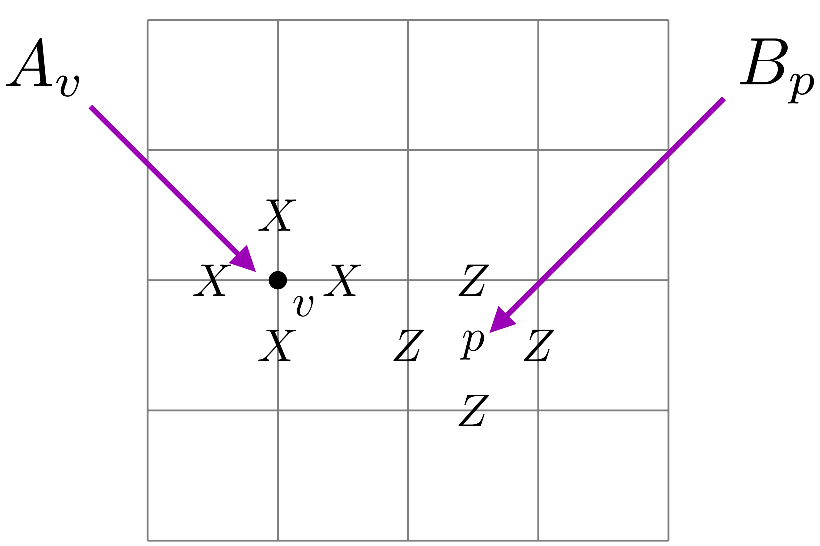
eigenstates of H besides ground states



For a string *L* of the lattice,

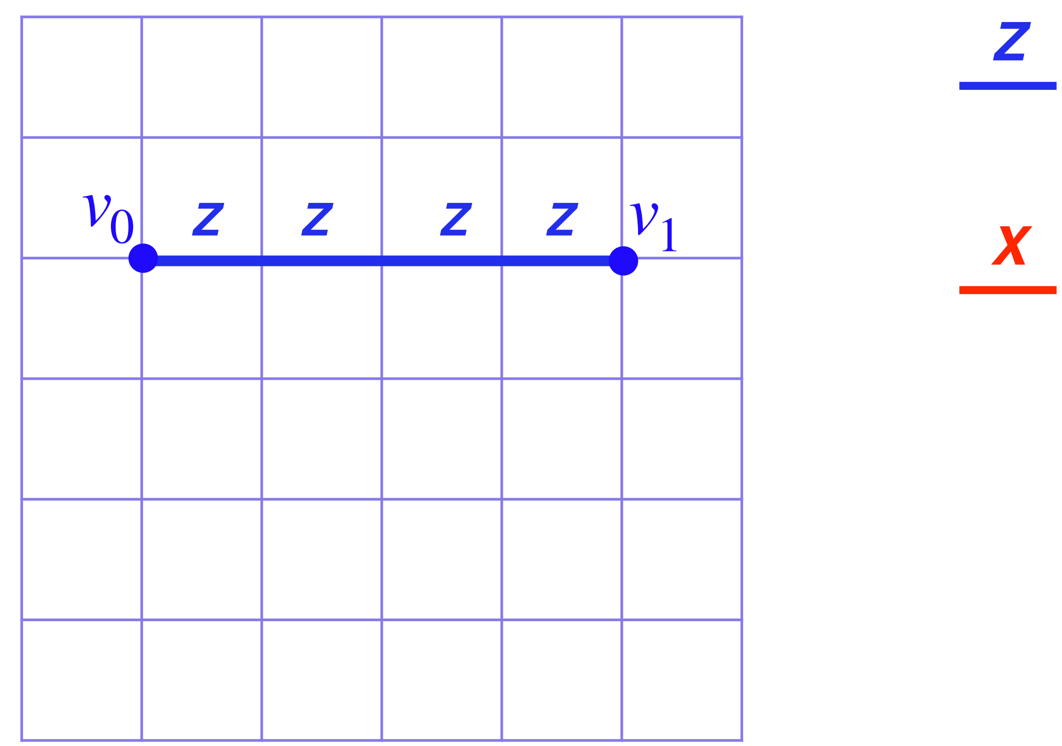
$$Z_L := \otimes_{e \in L} Z_e$$

- Z_L commutes with all B_p and all A_v except at the ends v_0, v_1 of *L*.



★Excited states:

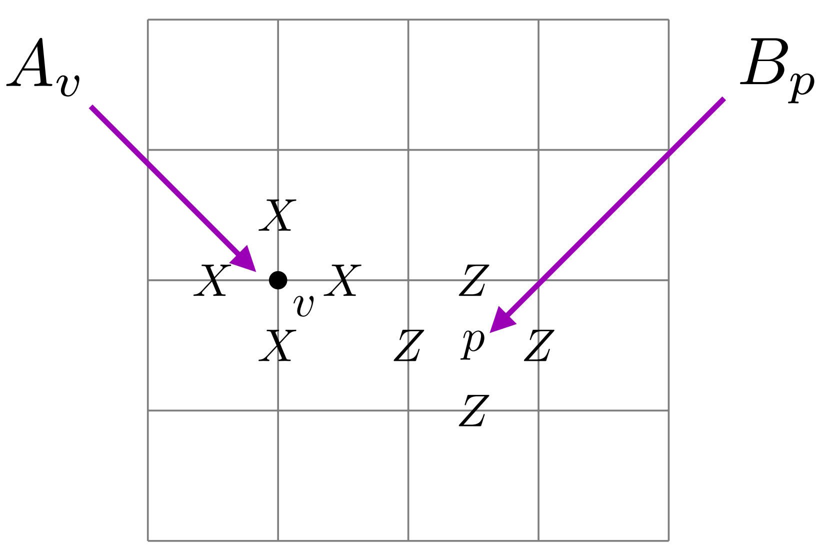
eigenstates of H besides ground states



For a string *L* of the lattice,

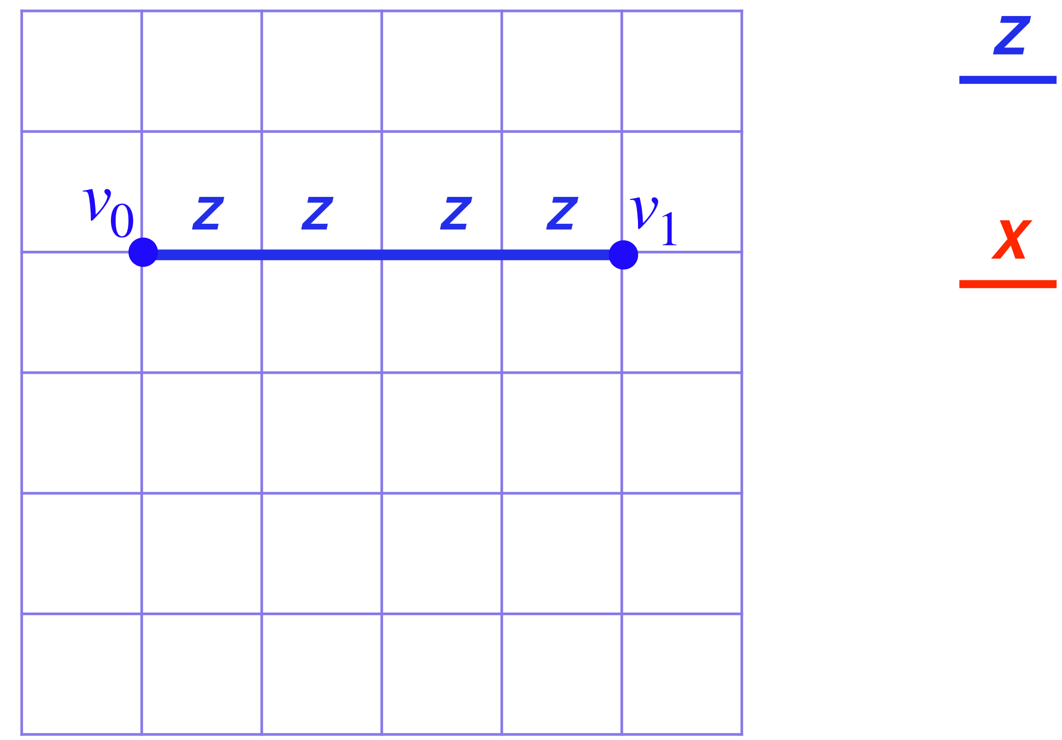
$$Z_L := \bigotimes_{e \in L} Z_e$$

- Z_L commutes with all B_p and all A_v except at the ends v_0, v_1 of *L*.
- Z_L anti-commutes with $A_{v_0}, \quad A_{v_1}$



★Excited states:

eigenstates of H besides ground states



For a string L of the lattice,

$$Z_L := \bigotimes_{e\in L} Z_e$$

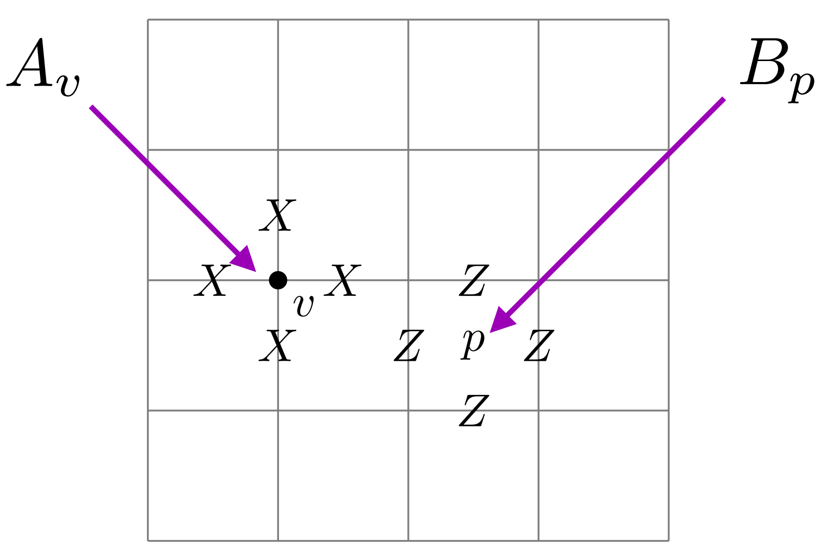
- Z_L commutes with all B_p and all A_v except at the ends v_0, v_1 of L.
- Z_L anti-commutes with A_{v_0}, $\quad A_{v_1}$

For $|\Omega\rangle \in V_{gs}$, let $|L\rangle := Z_L|\Omega\rangle$, $\Longrightarrow$

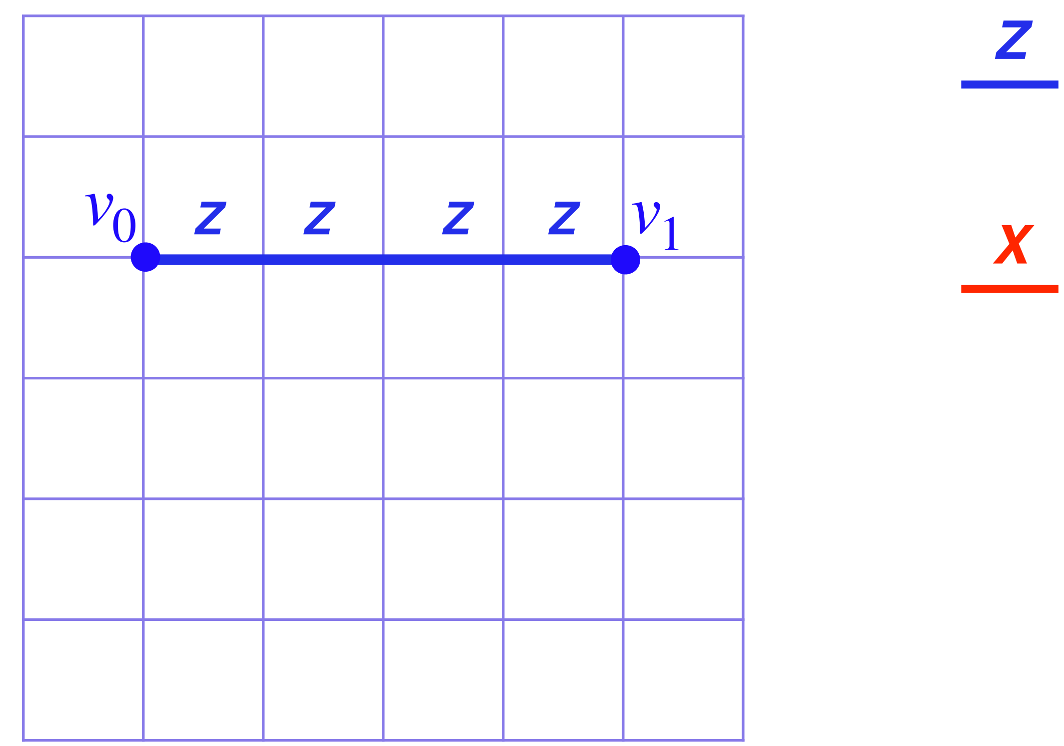
$$B_p|L\rangle = |L\rangle, \quad \forall p$$

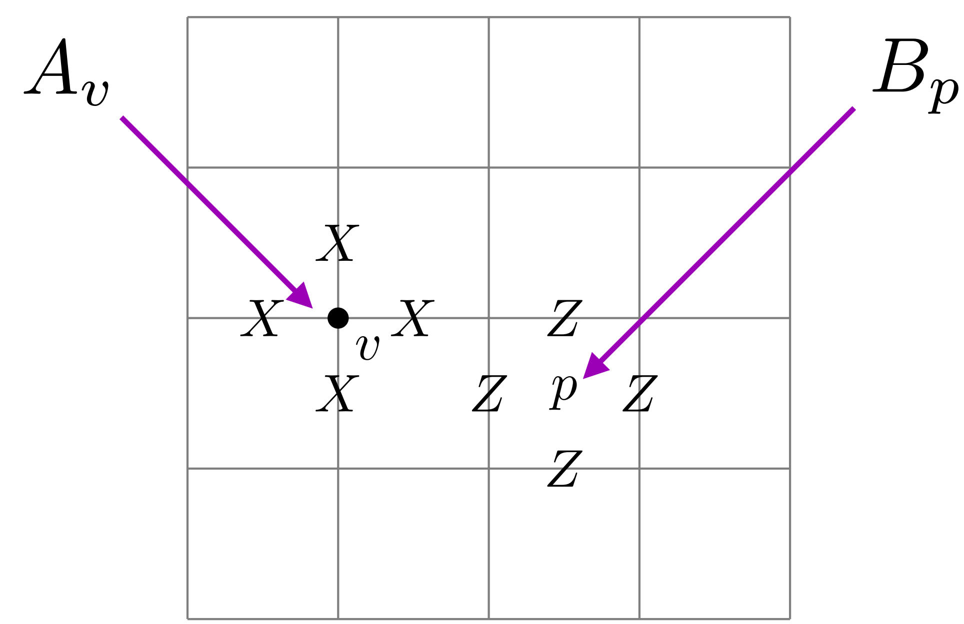
$$A_v|L\rangle = |L\rangle, \quad \forall v \neq v_0, v_1$$

$$A_{v_i}|L\rangle = -|L\rangle, \quad i = 0,1$$

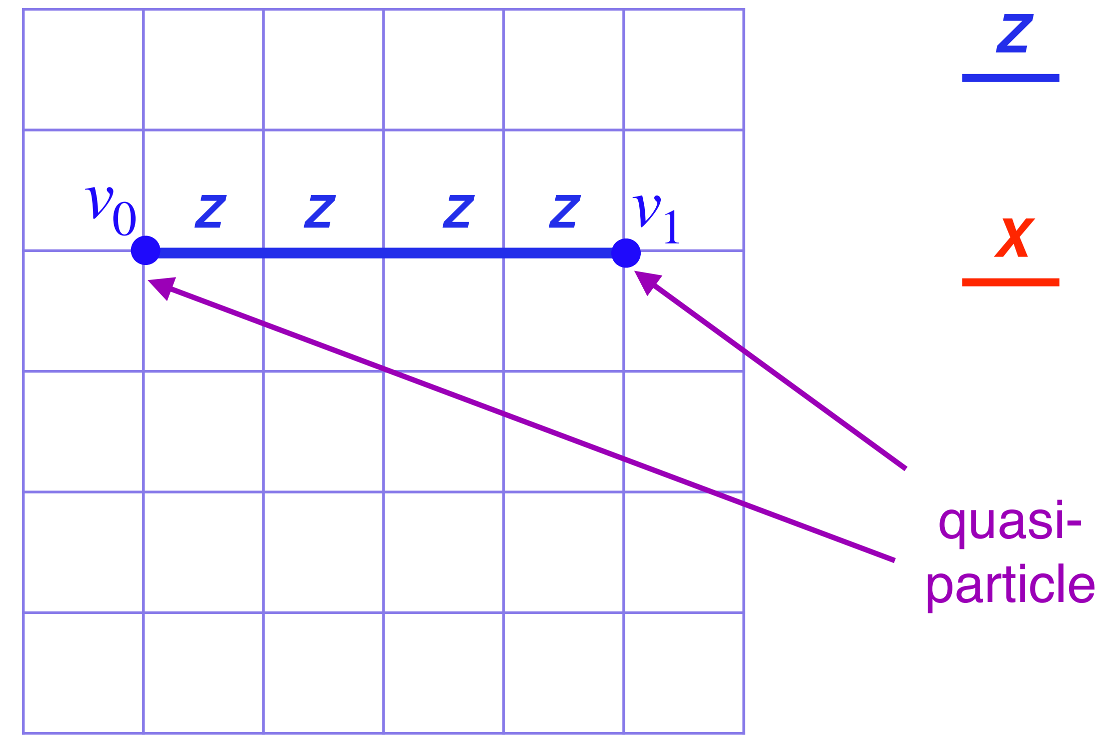


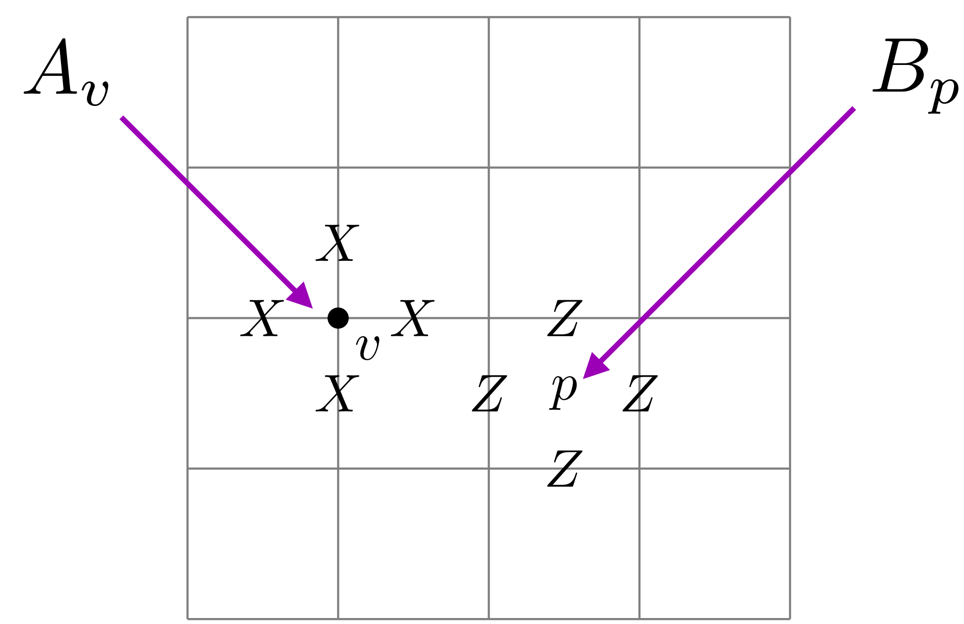
- $|L\rangle$ is excited state with eigenvalue/energy 4 more than that of the ground states;
- It represents a pair of quasi-particles at v_0, v_1.



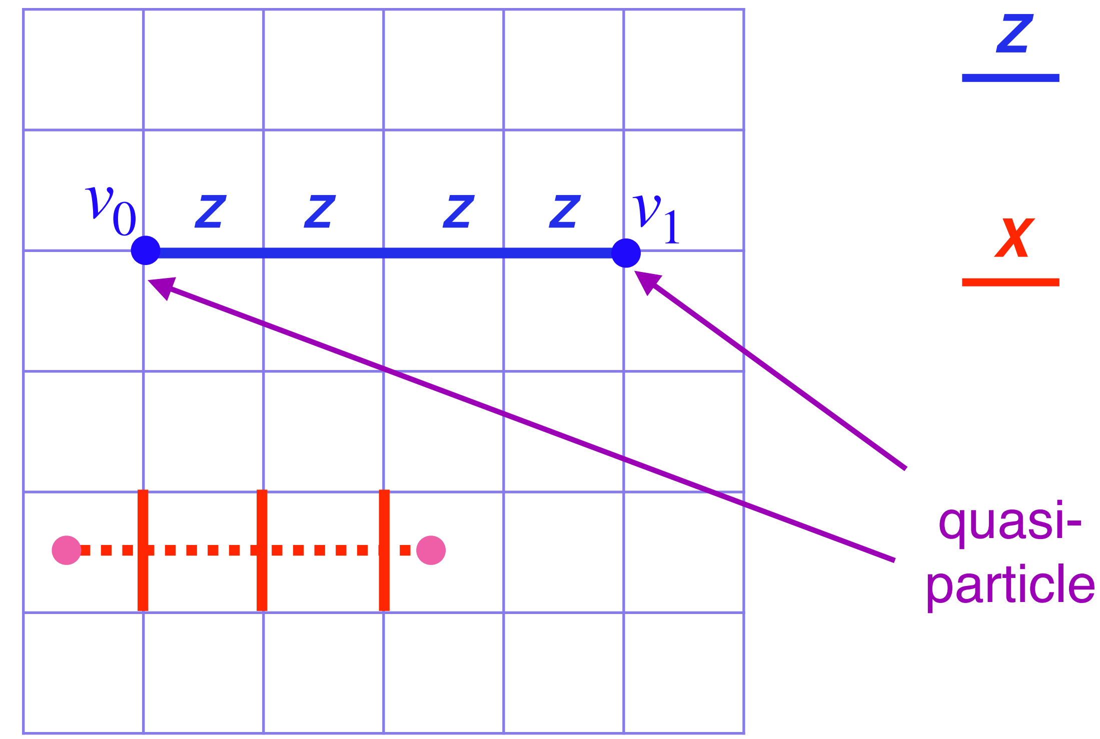


- $|L\rangle$ is excited state with eigenvalue/ energy 4 more than that of the ground states;
- It represents a pair of quasi-particles at v_0, v_1.





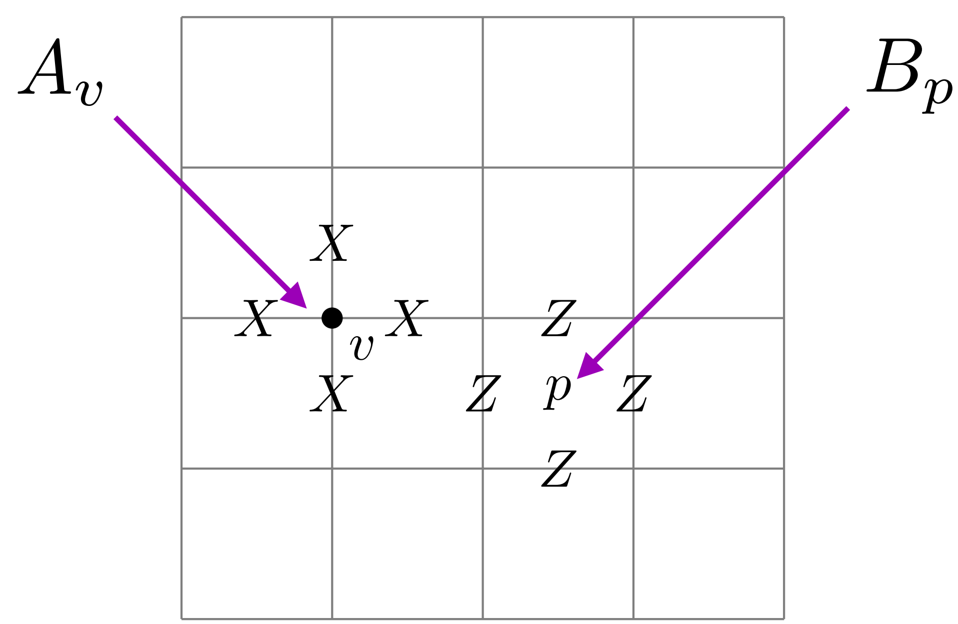
- $|L\rangle$ is excited state with eigenvalue/ energy 4 more than that of the ground states;
- It represents a pair of quasi-particles at v_0, v_1.



Similarly, if T is a string in the dual lattice,

$$X_T := \bigotimes_{e^* \in T} X_e$$

$X_T|\Omega\rangle$ is an excited state representing a pair of quasi-particles at the ends of T.

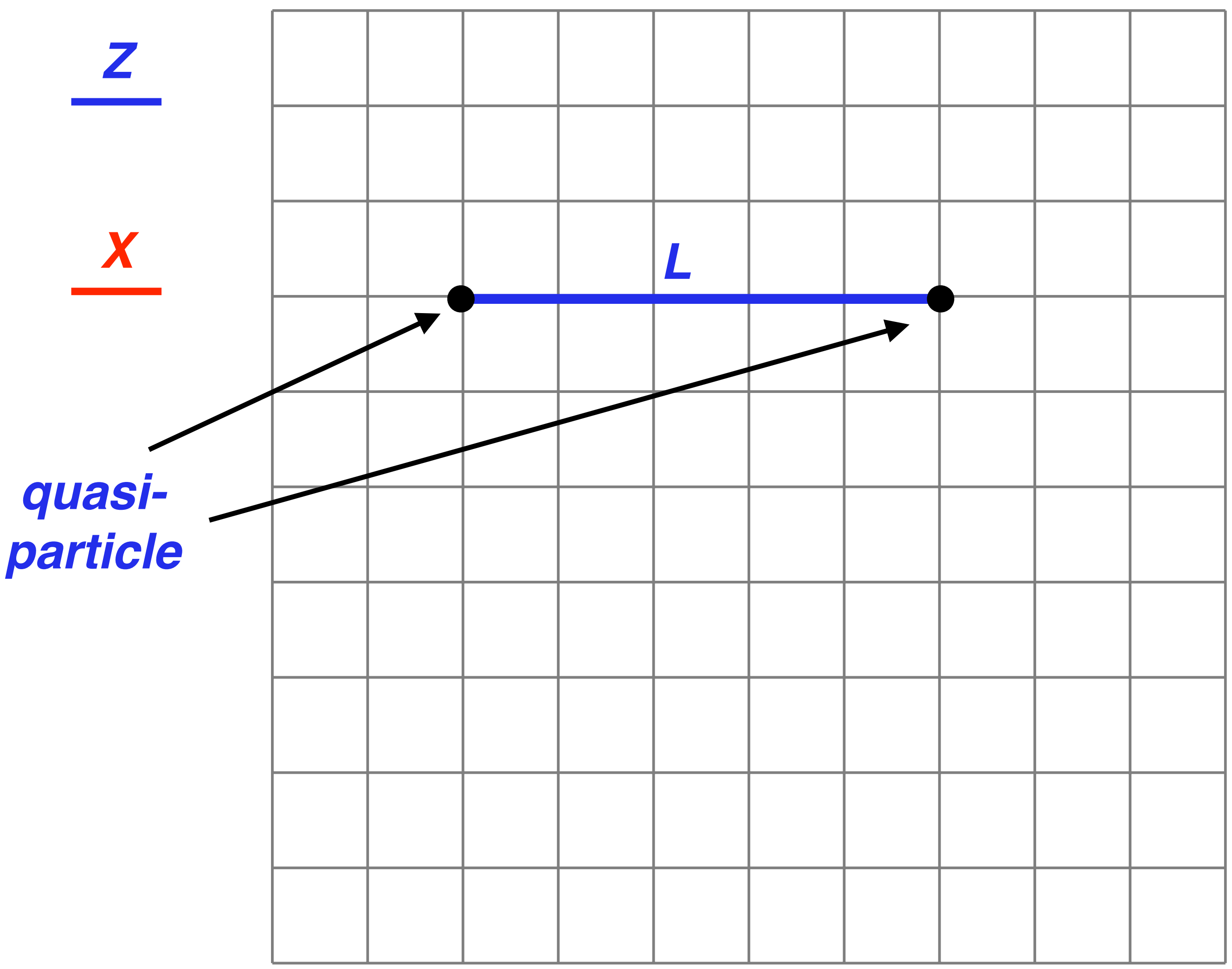


Braiding:

$$|\Omega\rangle \in V_{gs}$$

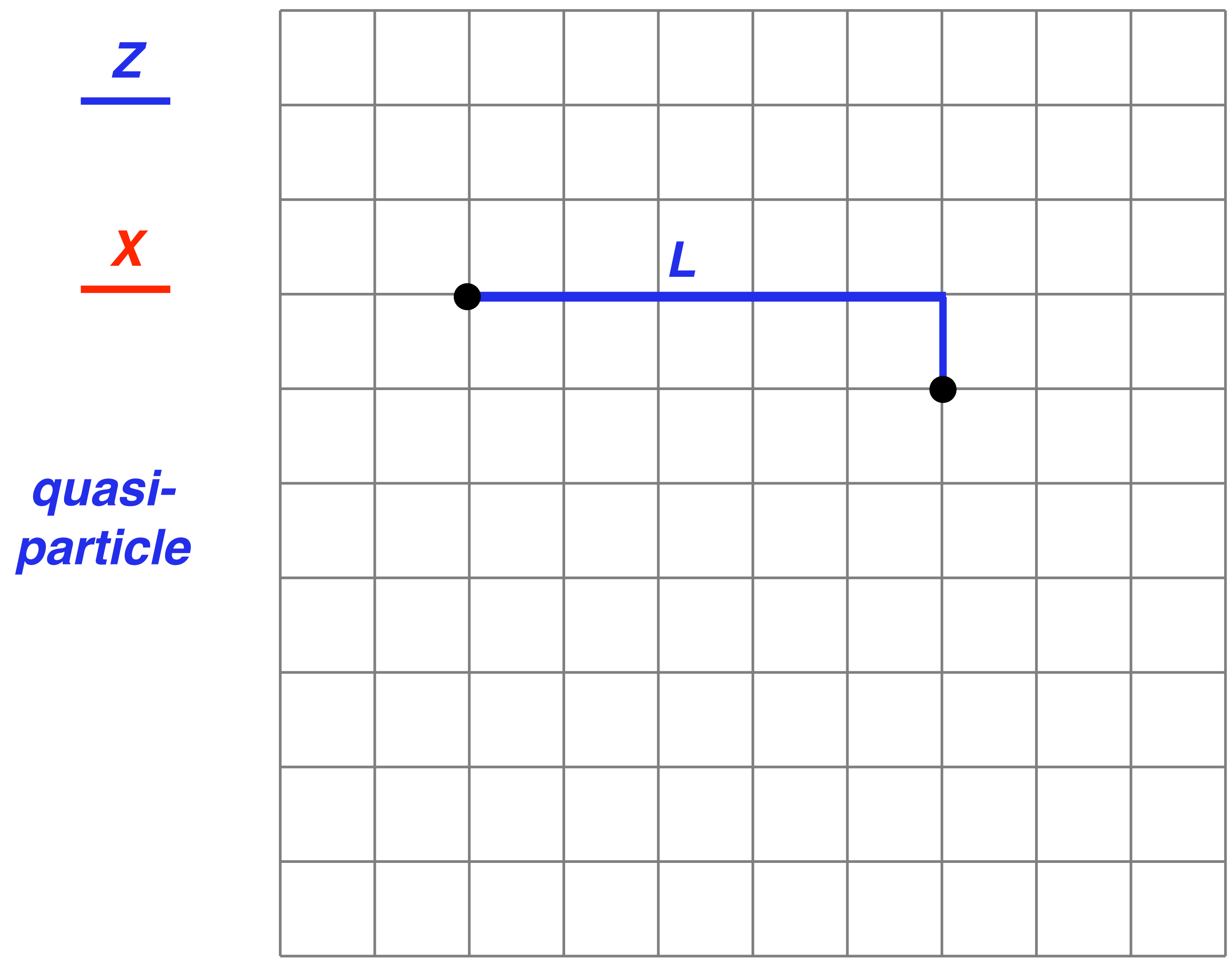
Braiding:

$$|\Omega\rangle \in V_{gs}$$



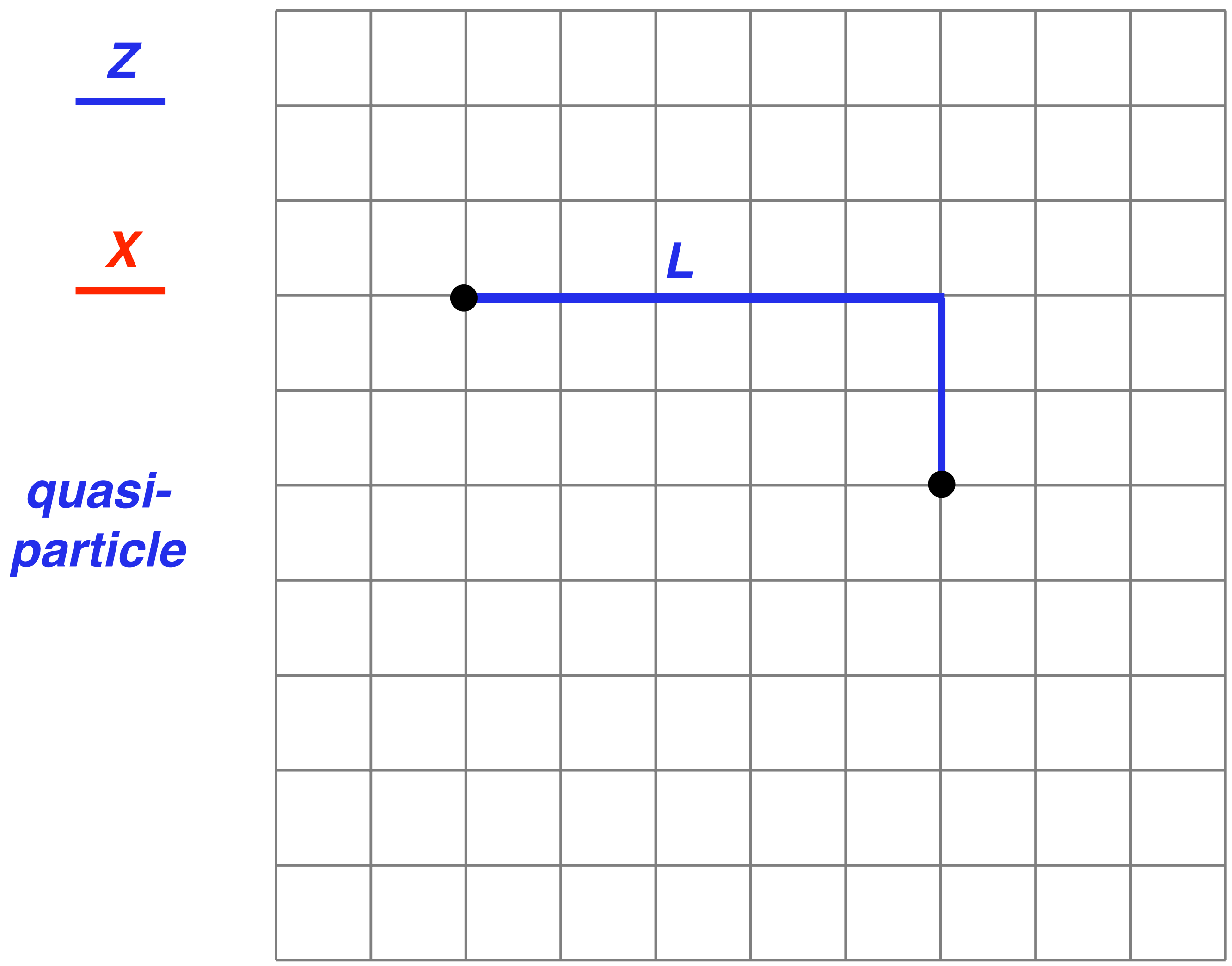
Braiding:

$$|\Omega\rangle \in V_{gs}$$



Braiding:

$$|\Omega\rangle \in V_{gs}$$

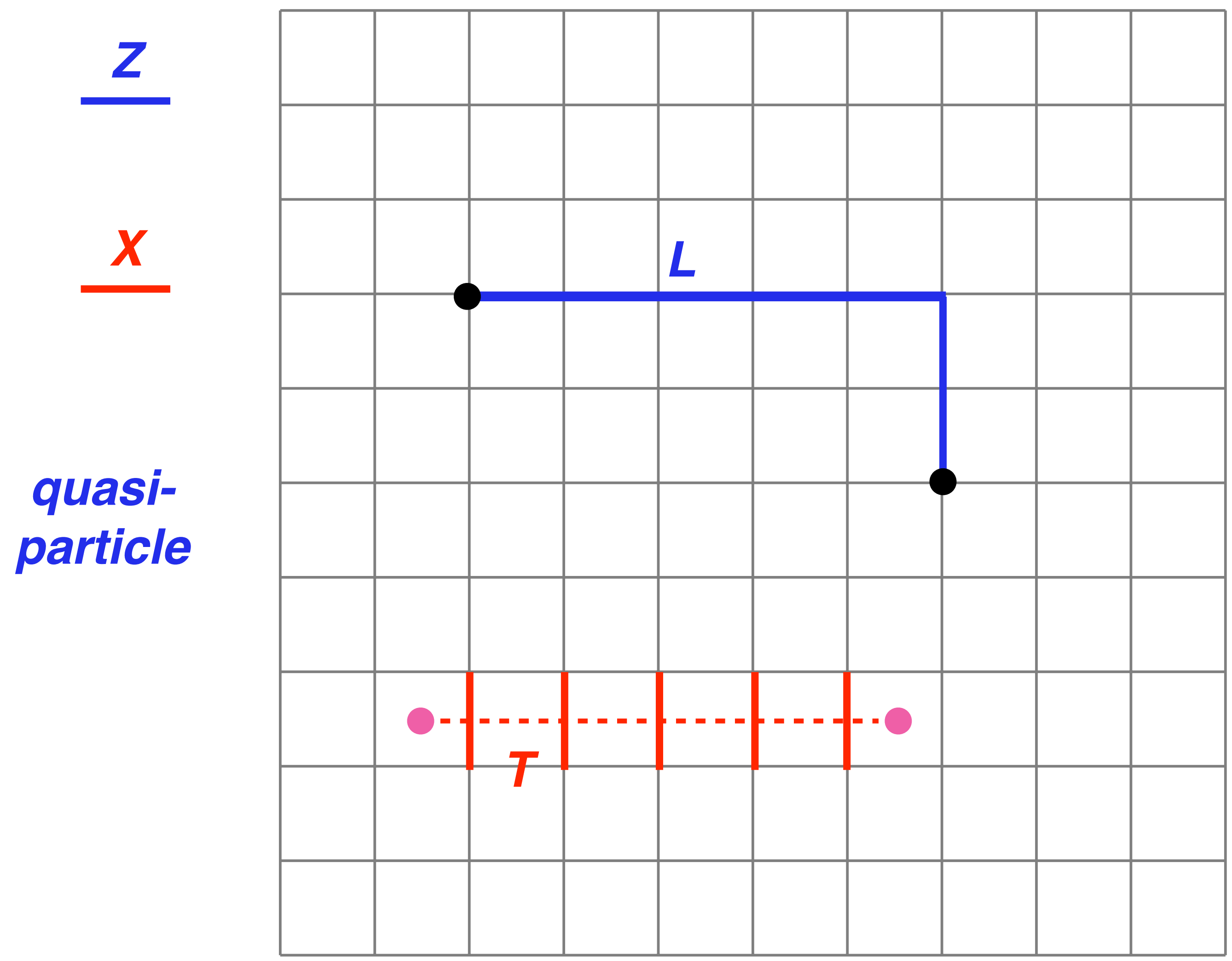


- Quasi-particles can move without cause energy.

Braiding:

$$|\Omega\rangle \in V_{gs}$$

$$|\psi\rangle_{in} = X_T\, Z_L\, |\Omega\rangle$$



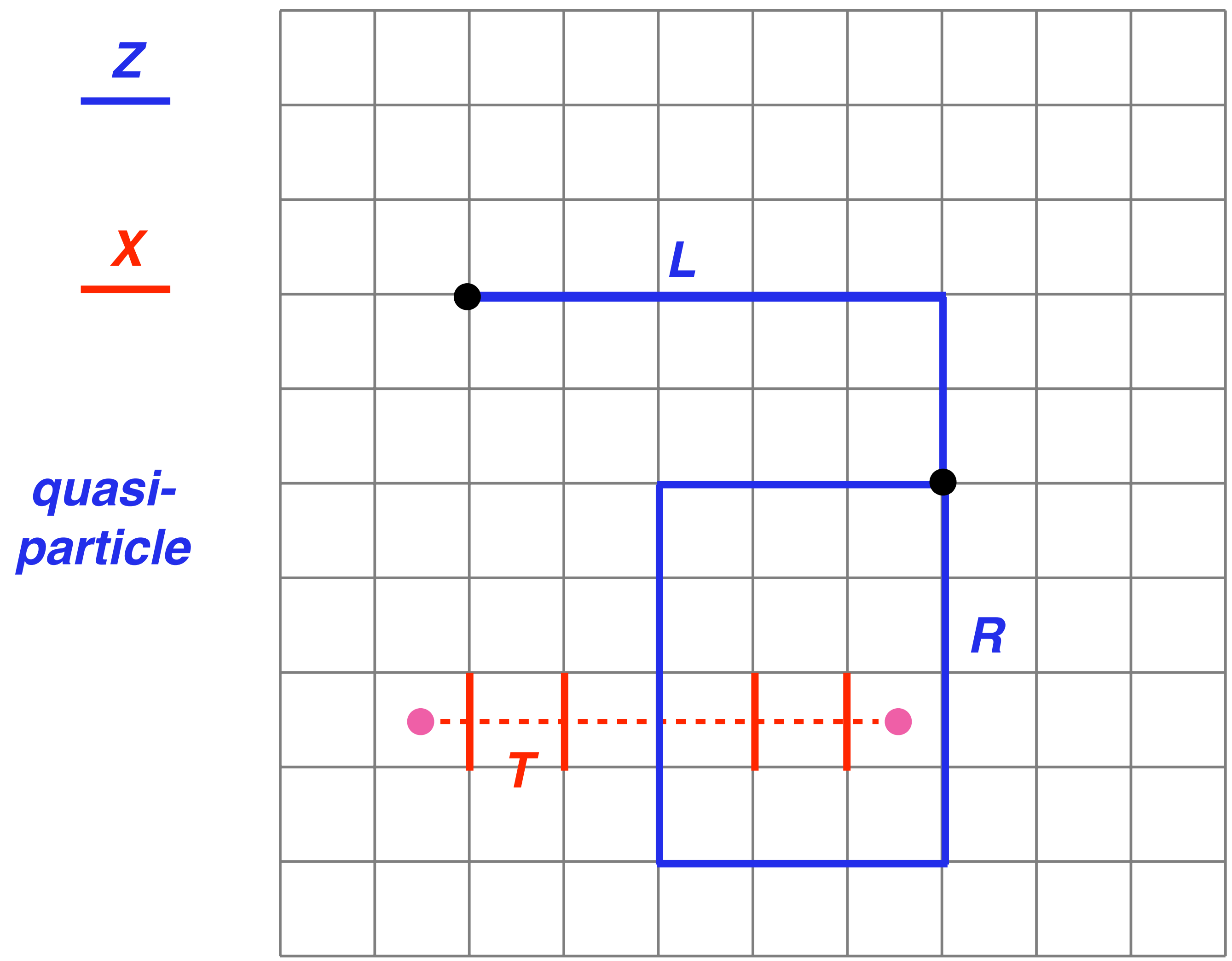
- Quasi-particles can move without cause energy.

Braiding:

$$|\Omega\rangle \in V_{gs}$$

$$|\psi\rangle_{in} = X_T\, Z_L\, |\Omega\rangle$$

$$\begin{aligned}
|\psi\rangle_{fi} &= Z_R X_T Z_L |\Omega\rangle \\
&= -X_T Z_R Z_L |\Omega\rangle \\
&= -X_T Z_L Z_R |\Omega\rangle \\
&= -X_T Z_L |\Omega\rangle \\
&= -|\psi\rangle_{in}
\end{aligned}$$



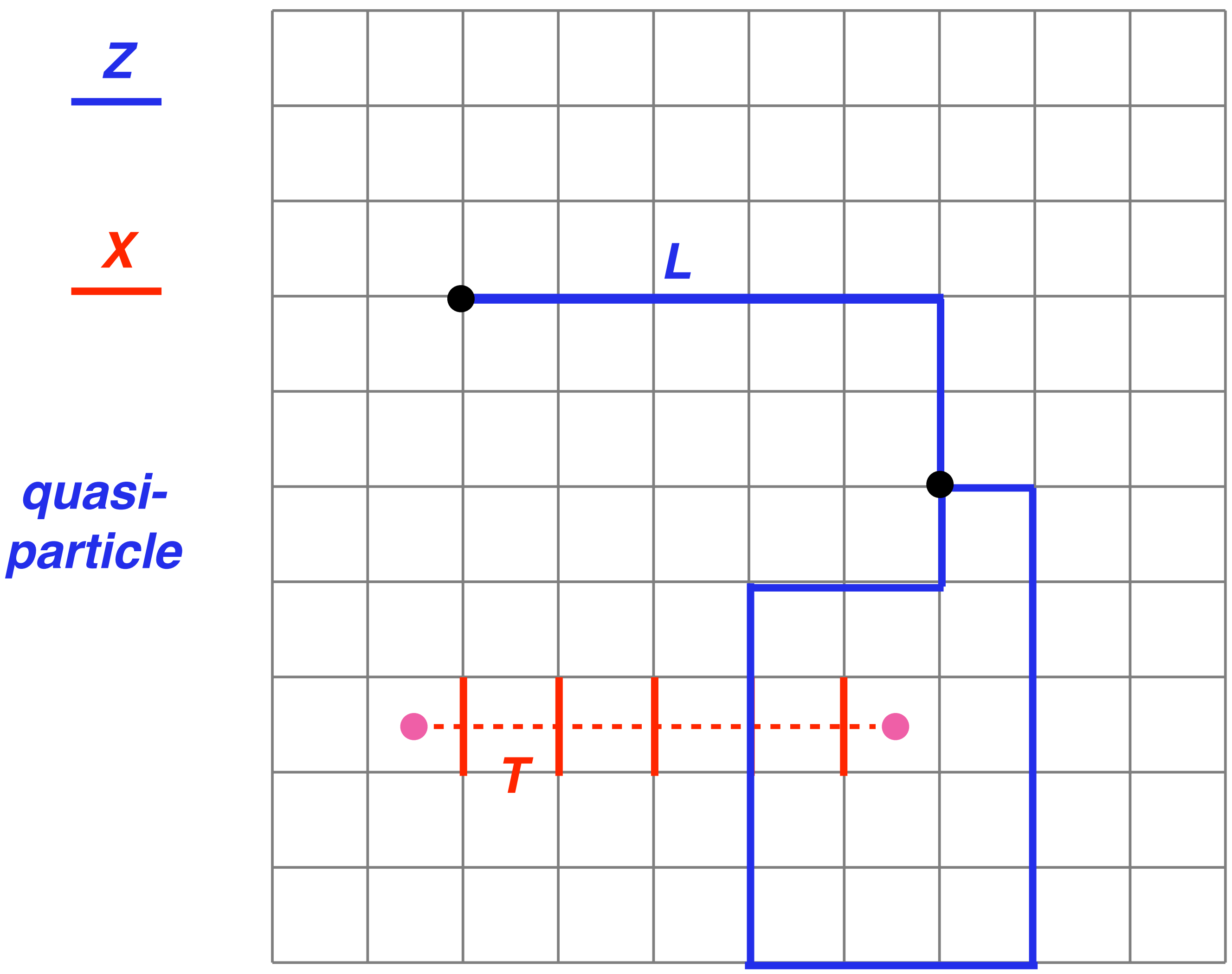
- Quasi-particles can move without cause energy.

Braiding:

$$|\Omega\rangle \in V_{gs}$$

$$|\psi\rangle_{in} = X_T \, Z_L \, |\Omega\rangle$$

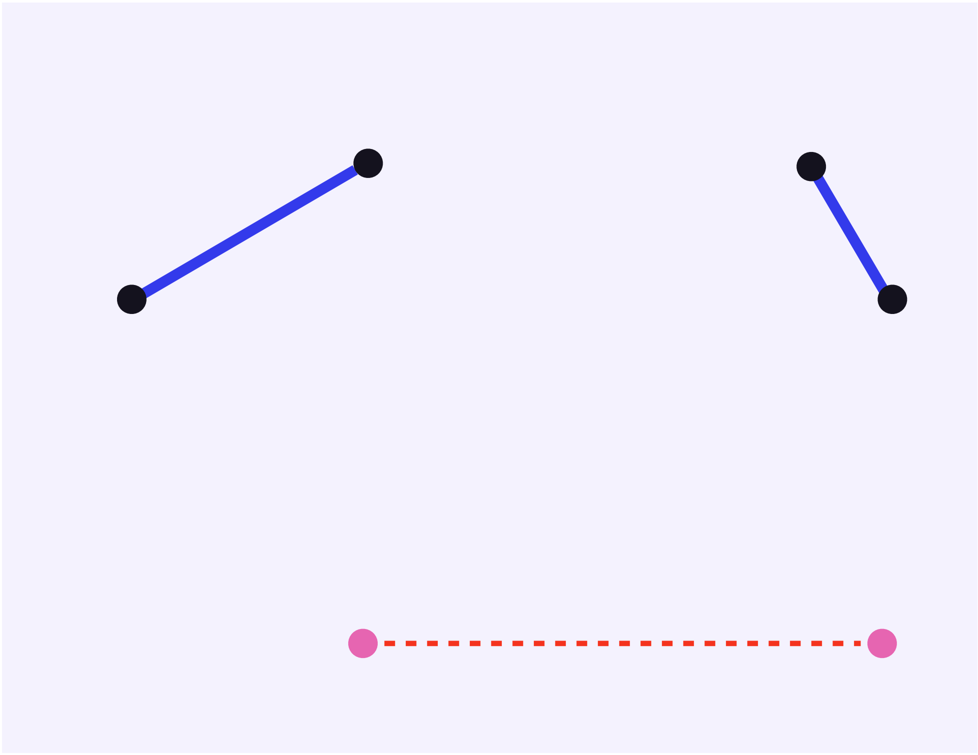
$$\begin{aligned} |\psi\rangle_{fi} &= Z_R X_T Z_L |\Omega\rangle \\ &= -X_T Z_R Z_L |\Omega\rangle \\ &= -X_T Z_L Z_R |\Omega\rangle \\ &= -X_T Z_L |\Omega\rangle \\ &= -|\psi\rangle_{in} \end{aligned}$$



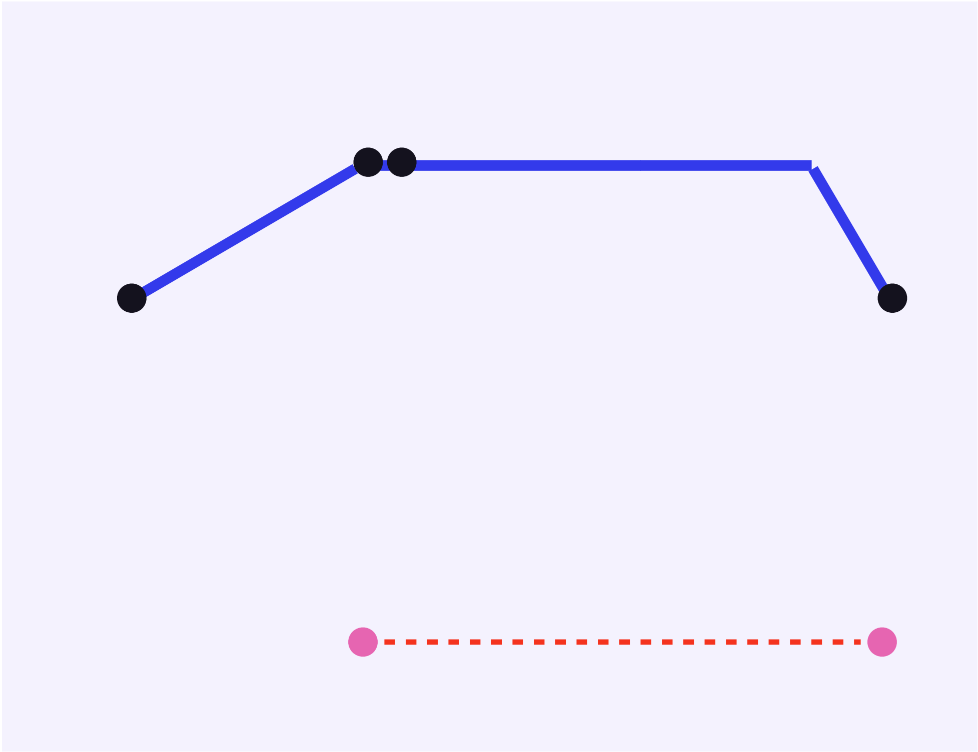
- Quasi-particles can move without cause energy.
- Braiding induces change of states invariant under isotopy.

Fusion:

Fusion:

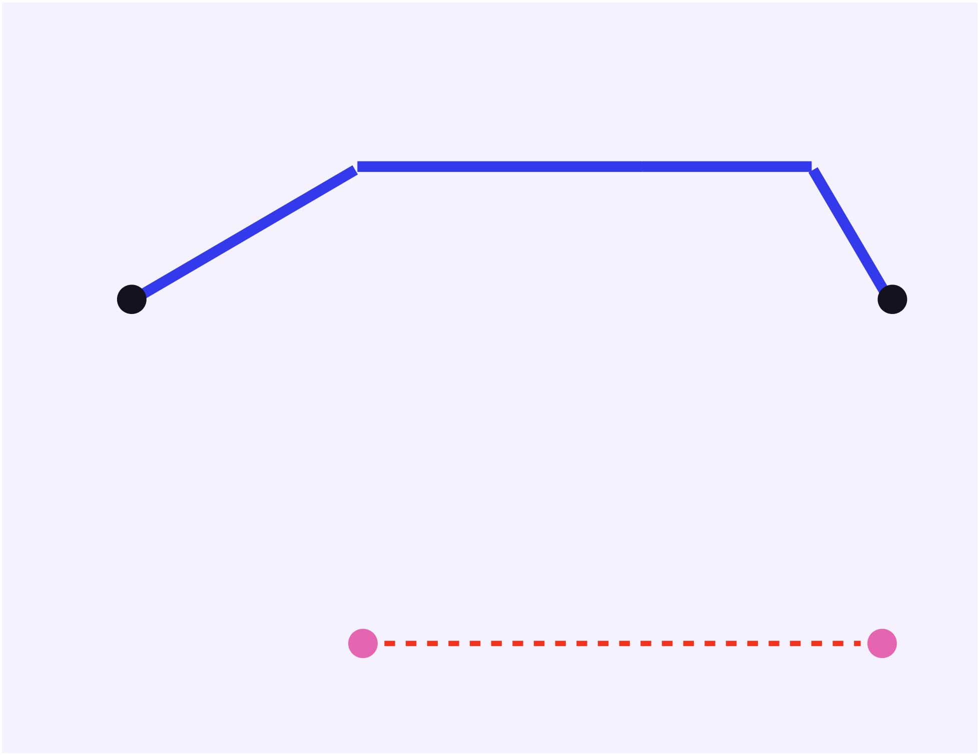


Fusion:



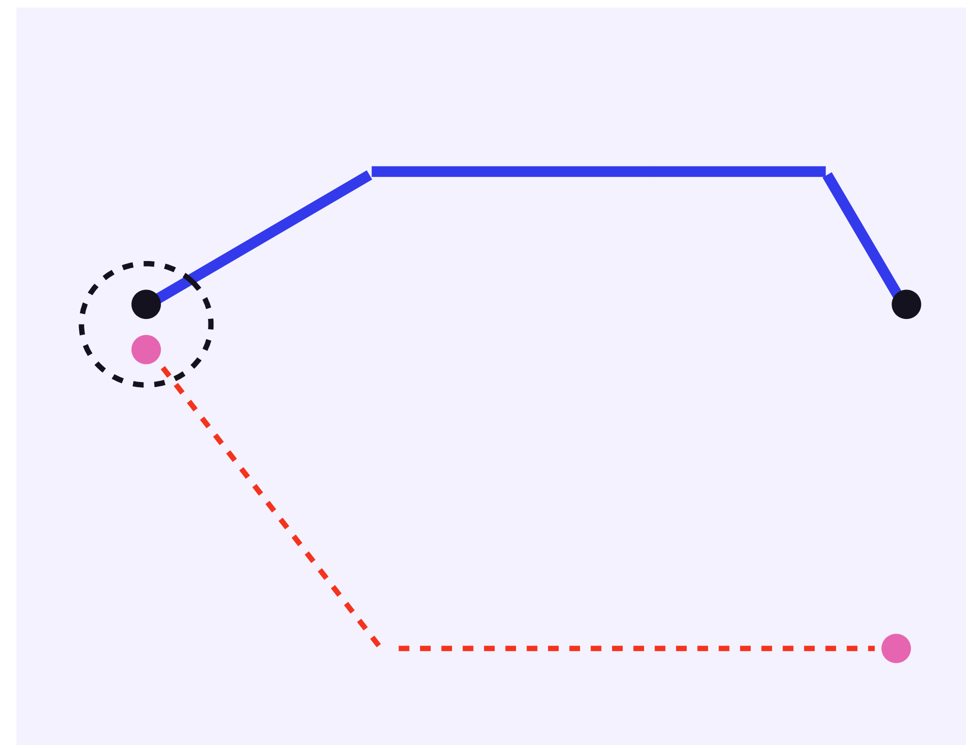
Fusion:

- Two • fuse to vacuum;
- Two • fuse to vacuum;
- • and • fuse to (• •) .



Fusion:

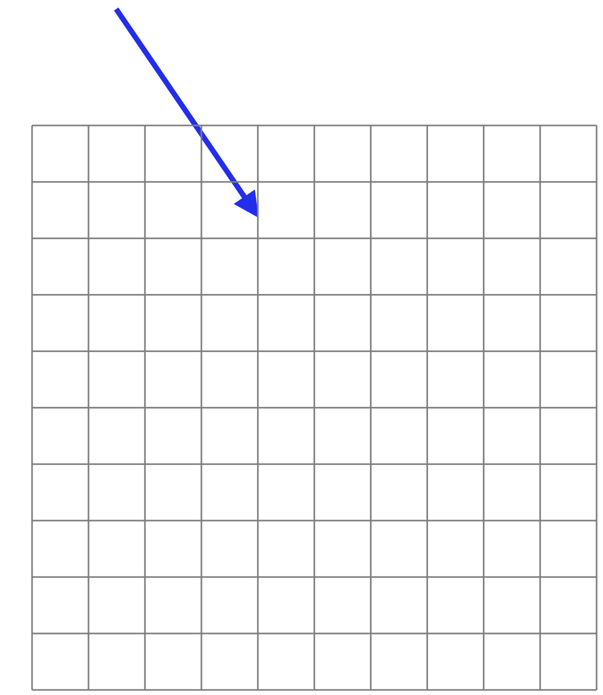
- Two ● fuse to vacuum;
- Two ● fuse to vacuum;
- ● and ● fuse to ● ● .



Kitaev quantum double model generalizing toric code

Fix a finite group $G = \{g_1, g_2, \cdots, g_d\}$

$\mathbb{C}[G] =$ **span**$\{\,|g_1\rangle, |g_2\rangle, \cdots, |g_d\rangle\}$



Total Hilbert/state space $\mathscr{H} = \bigotimes_{e \in E} (\mathbb{C}[G])_e$

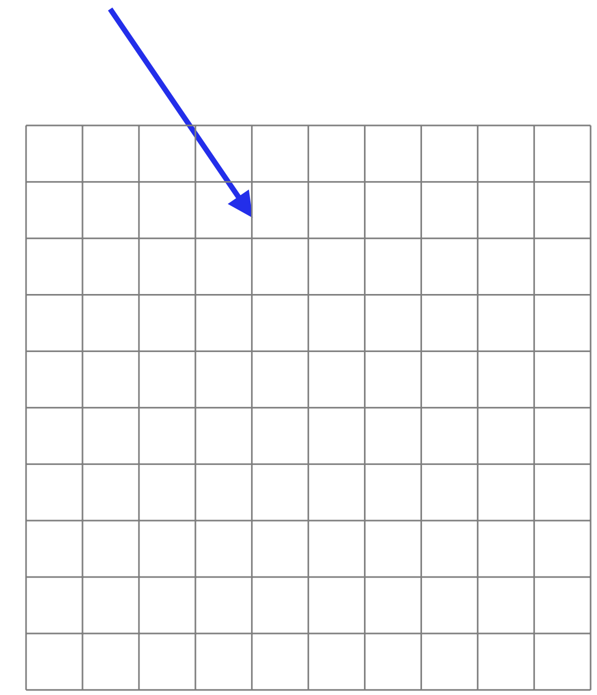
A standard basis of $\mathscr{H}$ consists of

$|s\rangle := \bigotimes_{e \in E} |s(e)\rangle$, for each $s : E \to G$

Kitaev quantum double model generalizing toric code

Fix a finite group $G = \{g_1, g_2, \cdots, g_d\}$

$\mathbb{C}[G] =$ **span**$\{\,|g_1\rangle, |g_2\rangle, \cdots, |g_d\rangle\}$



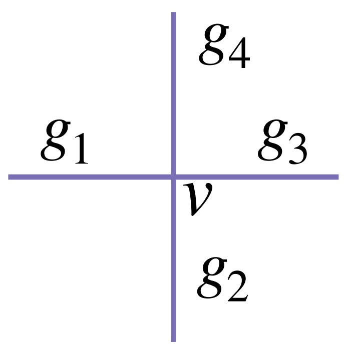
Total Hilbert/state space $\mathscr{H} = \otimes_{e\in E}\,(\mathbb{C}[G])_e$

A standard basis of $\mathscr{H}$ consists of

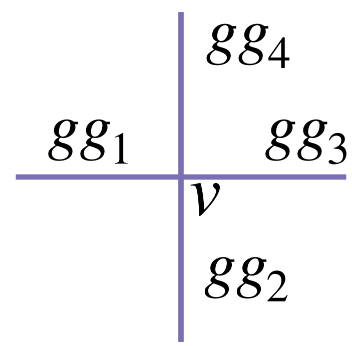
$|\,s\rangle \;:= \otimes_{e\in E}\,|\,s(e)\rangle$, for each $s : E \to G$

Define the Hamiltonian $\quad H = -\sum_{v\in V} A_v - \sum_{p\in F} B_p$

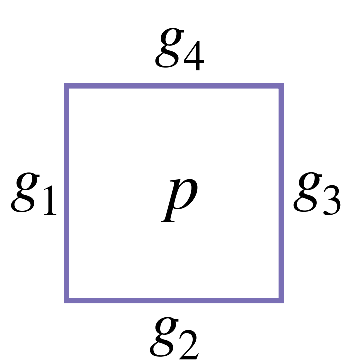
A_v



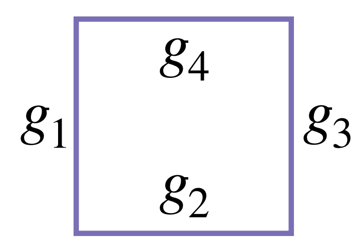
$= \quad \frac{1}{d}\sum_{g\in G}$



B_p



$= \delta_{g_1 g_2 g_3 g_4,\ id}$



The types of quasi-particle excitations are characterized by (C, χ)

C : a conjugacy class in G

χ : an irreducible representation of the centralizer of an element in C

$\boxed{\text{Irreducible representations of the quantum double of } G}$

The types of quasi-particle excitations are characterized by (C, χ)

C : a conjugacy class in G

χ : an irreducible representation of the centralizer of an element in C

$\boxed{\text{Irreducible representations of the quantum double of } G}$

Example: permutation group $S_3 = \{(1), (12), (23), (13), (123), (132)\}$

C	centralizer	χ	quasi-particle
$\{(1)\}$	S_3	$[+]$	A
		$[-]$	B
		$[2]$	C
$\{(12), (23), (13)\}$	$\mathbb{Z}/2\mathbb{Z}$	$[1]$	D
		$[-1]$	E
$\{(123), (132)\}$	$\mathbb{Z}/3\mathbb{Z}$	$[1]$	F
		$[\omega_3]$	G
		$[\bar{\omega}_3]$	H

Eight quasi-particle types

Fusion of quasi-particles

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

Fusion of quasi-particles

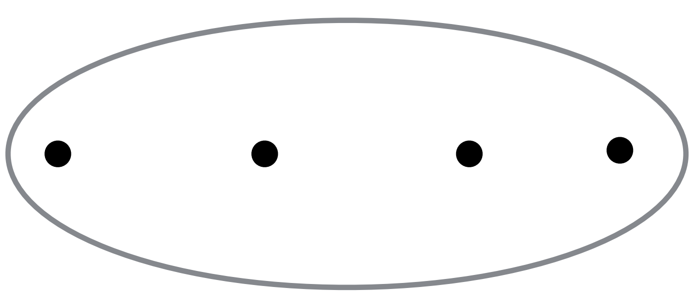
$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

- The fusion of two quasi-particles is a superposition of types;

Fusion of quasi-particles

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

- The fusion of two quasi-particles is a superposition of types;
- The state space of quasi-particles is degenerate even in the plane;

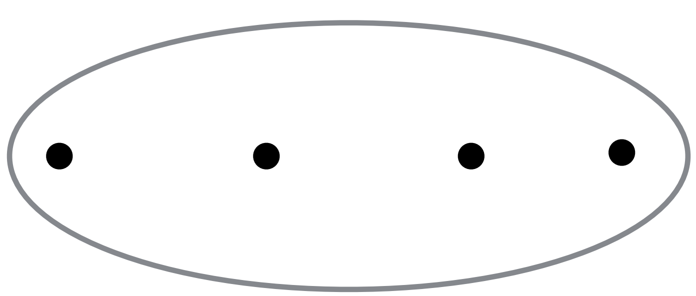


GSD > 1

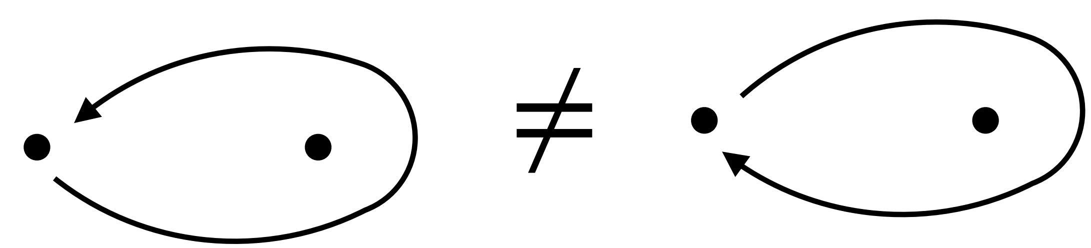
Fusion of quasi-particles

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

- The fusion of two quasi-particles is a superposition of types;
- The state space of quasi-particles is degenerate even in the plane;
- Non-Abelian Aharonov–Bohm effect.



GSD > 1



Unitary transformation

Some references:

- Kitaev, A. Yu. "Fault-tolerant quantum computation by anyons." Annals of physics 303.1 (2003): 2-30.
- Cui, Shawn X. "Topological quantum computation." Lecture Notes. https://www.math.purdue.edu/~cui177/Lecture_Combined.pdf
- Nayak, Chetan, et al. "Non-Abelian anyons and topological quantum computation." Reviews of Modern Physics 80.3 (2008): 1083-1159.
- Bonderson, Parsa. “Non-Abelian anyons and interferometry”. California Institute of Technology, 2012.

Thank You